WHAT IS A CURVE?

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1. Introduction. When the searching light of modern mathematical thinking is focused on the classical notion of a curve, this idea is found to involve elements of vagueness which must be clarified by accurate and exact definition. Fortunately this has been made possible and relatively simple by development in the field of set-theoretic topology. We shall endeavor to set forth below, first the need for explicit definition of a curve, then the definition itself, and finally several illustrations of types of simple curves which can be completely characterized by their topological properties and which more nearly approach the classical notion of a curve.

2. The classical notion. The concept of a curve as the "path (or locus) of a continuously moving point" usually is accompanied by intuitive notions of

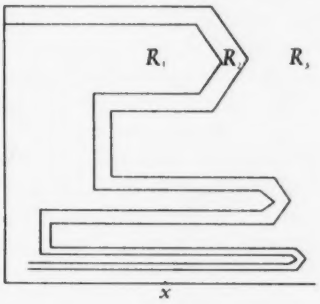


FIG. 1.

thinness and *two-sidedness*. When the curve is in a plane, these were thought to be consequences of the rather vaguely formulated definition of a curve as just given.

That the path of a continuously moving point is not necessarily a thin or curve-like set was shown by Peano and somewhat later by E. H. Moore, who demonstrated the remarkable fact that a square plus its interior can be exhibited as the continuous image of the interval. In other words, if S denotes a square plus its interior, we can define continuous functions $x(t)$ and $y(t)$ on the interval $0 \leqq t \leqq 1$ so that as t varies from 0 to 1, the point $P[x(t), y(t)]$ moves continuously through all the points of S.

A still more striking result in this direction is the remarkable theorem proved independently by Hahn and Mazurkiewicz about 1913. This theorem asserts that in order for a point set M (in euclidean space of any number of dimensions) to be representable as the continuous image of the interval $0 \leqq t \leqq 1$, it is necessary and sufficient that M be a locally connected continuum. (A *continuum* in euclidean space is a closed, bounded, and connected set; and a con-

tinuum M is *locally connected* provided that for any $\epsilon>0$ a $\delta>0$ exists such that any two points x and y of M at a distance apart $<\delta$ can be joined by a subcontinuum of M of diameter $<\epsilon$). Thus since obviously not only a square but also a cube, an n-dimensional interval, an n-dimensional sphere and a multitude of other sets are locally connected continua, any such set M can be represented as the path of a continuously moving point in the sense that we can define continuous functions

$$x_i = x_i(t) \qquad 0 \leqq t \leqq 1,\ i = 1, 2, \cdots, n,$$

such that as t varies from 0 to 1 the point P with coordinates $(x_1, x_2, \cdots, x_n)$ moves continuously through all the points of M.

Even when a set is sufficiently "thin" or "1-dimensional" that we would probably call it a curve it may be in a plane and still not be two-sided. To illustrate we note that in Figure 1 any point on the base of the continuum, such as x, is a boundary point of each of the three regions R_1, R_2, R_3 into which the continuum divides the plane. Hence there are *three sides* of the base of this continuum. (Clearly we could add extra oscillating curves to the figure so as to make an arbitrarily large number or even an infinite number of regions each having all base points x on their boundaries). Nevertheless our continuum is a thin 1-dimensional set made up of an infinite number of line segments. Now it is possible to construct in a plane a continuum which is thin in the sense that it will not contain the interior of any circle and yet is so unusual that it will divide the plane into any finite number or an infinite number of regions and, further, it will be the boundary of each one of these regions. Also a plane continuum can be constructed which not only itself cuts the plane into infinitely many regions but has the remarkable property that every subcontinuum of it (any "piece" of it) also cuts the plane into infinitely many regions.

3. Dimensionality. General definitions of curve, surface, solid. Undoubtedly sufficient evidence has been given of the necessity of being precise in our definitions and statements concerning curves, surfaces, etc., and of the unreliability of our intuition concerning these concepts.

We leave aside the continuous traversibility of the set as a criterion characterizing or distinguishing between curves, surfaces, solids, etc., since we have seen how it fails in this respect, and concentrate on content or dimensionality of the set as a guide.

Hence it seems natural and adequate to define a *curve* as a 1-dimensional continuum, a *surface* as a 2-dimensional continuum and a *solid body* as a 3-dimensional continuum.

These definitions are satisfactory provided we give an adequate definition of dimensionality of a set. To this end let us concentrate our attention on compact sets, *i.e.*, sets K which have the property that any infinite subset has a limit point belonging to K, sets which are closed and bounded if they lie in a euclidean space.

We then define the dimensionality of the empty set to be -1 and agree the dimensionality of any other set is to be $\geqq 0$. Assuming, then, that we have defined the dimensionality concept for dimensions $\leqq n-1$, by induction we define a set K to be of dimensionality n provided (1) every pair of distinct points p and q of K can be separated in K by some set X of dimensionality $\leqq n-1$, *i.e.*, $K-X$ falls into two separated sets K_p and K_q containing p and q respectively; and (2) some pair of points of K cannot be separated in K by a subset of K of dimensionality $<n-1$. Thus for $n \geqq 0$, a set K is of dimension n provided n is the least integer such that every pair of distinct points of K can be separated in K by the removal of a subset of dimension not greater than $n-1$.

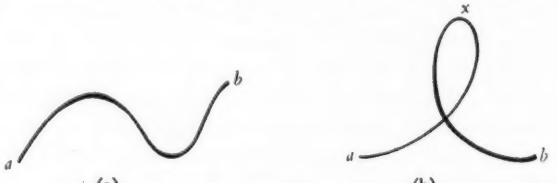
According to this definition, then, a compact set K is of dimension 0 provided every two points of K can be separated in K by omitting the empty set, *i.e.*, provided they are already separated in K. Hence a 0-dimensional set is one which is non-empty but is totally disconnected in the sense that its only connected subsets are single points. A compact set K is 1-dimensional provided any two points can be separated in K by omitting from K a 0-dimensional or totally disconnected set but some two points cannot be separated without omitting some points from K. A compact set K is 2-dimensional provided each pair of points of K can be separated in K by omitting a 1-dimensional set but not every pair can be separated by omitting a 0-dimensional set, and so on.

Stated in other terms, if we accept our definition that a curve is a 1-dimensional continuum, a surface is a 2-dimensional continuum, and a solid body is a 3-dimensional continuum, we see that a non-empty compact set K is 0-dimensional if every pair of its points are separated in K. The set is 1-dimensional at most provided we can (with shears if you like) separate any two of its points by cutting the set along a 0-dimensional set, *i.e.*, by cutting out only single points as connected pieces. The set is 2-dimensional at most provided we can separate any two points by cutting the set along a 1-dimensional set, *i.e.*, by cutting out only curves as connected sets. The set is 3-dimensional at most if we can separate any two points by cutting (with a saw perhaps) the set along a 2-dimensional set, *i.e.*, by cutting out only surfaces as connected sets.

4. Some simple types of curves. Having defined exactly the notions of curve, surface, and solid in terms of their topological properties in such a way that they correspond roughly to the geometrical notions of line, plane, and space, we consider now some interesting particular kinds of curves which may be similarly characterized.

Take first a straight line interval ab joining two points a and b and ask the question "What properties of a set make it essentially like an interval?" or "When are the points in a set associated together like those in the interval ab?" For example, if ab is a taut string and we release the tension and let it go slack but do not allow it to loop over onto itself, it is no longer straight but it retains its same essential structure. It can still be severed by cutting out any one of its points other than a or b; and it is this property in particular which char-

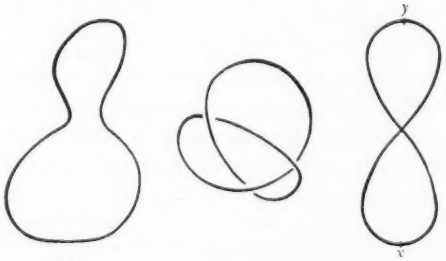
acterizes the interval completely from the topological point of view. In other words, if we understand by a *simple arc* any set of points which is topologically equivalent to an interval in the sense that its points can be put into one-to-one and continuous correspondence with the points of an interval, then *in order that a continuum T be a simple arc it is necessary and sufficient that T contain two points a and b such that the removal of any point of T other than a or b will disconnect T.* Thus in Fig. 2, (a) is a simple arc, but (b) is not a simple arc because the removal of neither a, b, nor x will separate the set (*i.e.*, will make it fall apart).



(a) (b)

FIG. 2.

Consider next a circle C and let us ask similar questions. If C is distorted, as was our interval, by letting it slacken and bend but not fold onto itself or be broken violently, it is seen to retain its essential set structure. It retains the property, for example, of being severed by the removal of any two of its points whatever. Here again the property mentioned is characteristic for the type of curves which are topologically equivalent to the circle. In other words, if we define a *simple closed curve* as a set which can be put in one-to-one and continuous correspondence with a circle, then *in order that a continuum C be a simple closed curve it is necessary and sufficient that C be disconnected by the omission of any two of its points.* Thus in Figure 2, (a) is not a simple closed curve since the removal of both a and b leaves the set connected. In Figure 3, (a) and (b) are simple closed curves but (c) is not a simple closed curve because the removal of x and y leaves the set connected.

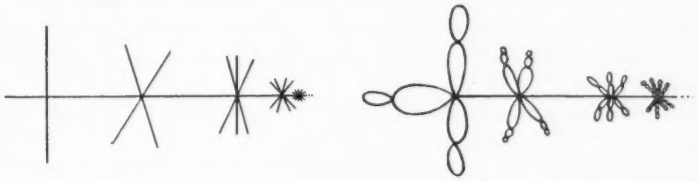


(a) (b) (c)

FIG. 3.

A curve which is made up of a finite number of simple arcs which overlap with each other only at end points of themselves is called a *graph* or a *linear*

graph. A graph, then could be regarded as being constructed by putting together in any one of numerous ways a finite number of simple arcs so that no two of the arcs will overlap anywhere except possibly at an end point of both. All of the curves illustrated in Figs. 2 and 3 are graphs; and of course many more complicated structures could be made which would still be graphs. However, if a graph is in a plane it, like the simpler curves previously discussed, will have the classical property of 2-sidedness which does not belong to all curves.



(a) (b)

FIG. 4.

Finally, we mention two further types of curves which in general are not graphs and yet whose structure is interesting and simple, namely the *dendrite* or *acyclic curve* and the *boundary curve*. A *dendrite* is a locally connected continuum which contains no simple closed curve. It may contain infinitely many simple arcs [See Fig. 4 (a)]. In fact it may be impossible to express it as the sum even of countably many arcs, and yet it has the property that any two of its points are end points of one and only one arc in the curve. A *boundary curve* is a locally connected continuum which can be so imbedded in a plane that it will be the boundary of a connected region of the plane. Although it is true that every dendrite is a boundary curve, in general a boundary curve will contain one and may contain infinitely many simple closed curves [See Fig. 4 (b)]. However, it is interesting to note that no such curve could contain a cross bar on a simple closed curve. In other words, the most that any two simple closed curves can overlap is in a single point (point of "tangency"). Thus any boundary curve breaks up into so called cyclic elements which are either single points or simple closed curves, no two of these have more than one common point, and these fit together to make up the curve and give it a structure relative to these elements which is very similar to that of a dendrite. [Compare Fig. 4 (a) with Fig. 4 (b)].

5. Conclusion. We have touched but a few of the many interesting aspects of the fundamental theory of curves. The subject has an extensive literature, particularly from the topological point of view, which the explorative reader will find fascinating as well as instructive. The field is a live one and it is currently receiving important contributions. Interesting and difficult problems remain unsolved. There is much to attract and repay the student who will expend the effort necessary to acquire a knowledge of these problems and to master the methods which have been devised for attacking them.

ON THE TEACHING OF ELEMENTARY MATHEMATICS*

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The following remarks are addressed to those who are preparing themselves to be teachers of elementary mathematics. They begin with some horrible examples of what I think a good teacher should not do, and some general principles that I think a good teacher would do well to follow; they conclude with a serious thesis concerning the proper emphasis of various aspects of elementary mathematics.

Most of the examples come out of my own first-hand experience at various institutions (although some do not) and could be, but will not be, documented. If this be undiplomatic, it is at least not appeasement. I shall not discuss what should be taught to freshmen because this depends to some extent on what they are being prepared for, and because I have expressed my views fully on this subject elsewhere. Rather, I shall confine myself to remarks which I consider to be valid regardless of what topics the curriculum contains.

There are at least three different French languages: Parisian French, Gascon French, and a language spoken by American tourists called High School French. In our subject, we have Pure Mathematics, Applied Mathematics, and a peculiar subject sometimes called "Freshman Mathematics." Every instructor, in the normal exercise of his duties, accumulates his own collection of specimens of "Freshman Mathematics." An innocuous example is that of the student who divided a large number by 1 by the process of long division. I call this innocuous because at least he was using a general algorithm which could be applied, however inadvisedly, to his problem. A more common and harmful sample is the following:

$$\frac{2+x}{2x} = \frac{\overset{1}{\cancel{2}}+\overset{1}{\cancel{x}}}{\underset{}{\cancel{2}\cancel{x}}} = 1 + 1 = 2. \tag{1}$$

This illustrates the "law of universal cancellation" which says that whenever a symbol occurs twice on the same paper or blackboard it may be crossed out in both places. A well-known "proof" of it proceeds as follows:

$$\frac{16}{64} = \frac{166}{664} = \frac{1666}{6664} = \cdots = \frac{1}{4}.$$

Very common specimens of "Freshman Mathematics" are the following:

$$\sin(x+y) = \sin x + \sin y, \qquad \log(x+y) = \log x + \log y, \tag{2}$$

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}.$$

* Based on an address delivered at the triennial convention of Pi Mu Epsilon held at Lehigh University, January 1, 1942.

Other common errors are "transposing" $2x=1$ to obtain

$$x = \frac{1}{-2} \tag{3}$$

(presumably because whenever a symbol is "brought" from one side of an equation to the other its sign must be changed); and adding fractions by adding numerators and denominators separately as in

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}. \tag{4}$$

A favorite specimen of mine is the following "proof" of the remainder theorem:

$$\begin{array}{r|l} x-a & f(x) \quad \Big| f \\ \hline & f(x)-f(a) \\ \hline & f(a). \end{array}$$

A young teacher may well feel at times as though he has somehow blundered into a psychopathic ward. But successful psychoanalytic technique requires sympathy, understanding of the difficulties, and probing into the past of the patient to find the events leading up to the tragedy. I can speak at first hand of this latter aspect since I have taught both high school and college freshman classes for many years.

It is essential for the teacher to understand both the student's preparation and difficulties, and, if I may be so bold as to say so, the subject he is teaching. I emphasize this because I once heard a man high in educational circles assert that one who knows too much can't be a good teacher. This is unquestionably true, but only because one who knows too much cannot exist. Of course, there are distinguished scholars who do not teach well, and excellent teachers who are not distinguished scholars. But if one has the patience and sympathy needed for good teaching and is willing to devote time and thought to the problems of teaching, then increased scholarship can only enrich one's teaching, not spoil it. On the other hand it is impossible for any one to teach well something he doesn't himself understand. There are, I am afraid, some people teaching the third reader who have never progressed beyond the fourth. By this I do not mean at all that it is necessary for a teacher of elementary subjects to be well versed in partial differential equations or the calculus of variations in the large, or similar advanced technical subjects; but I do think it essential that he have a broad, deep, and clear understanding of the fundamental concepts underlying elementary mathematics. That such understanding is not always present becomes obvious upon contact with some teachers and some textbooks.

For example, I was once asked to settle a bet between two ex-colleagues, one of whom contended that $0/0=0$ because 0 divided by anything is 0, while the other held that $0/0=1$ because anything divided by itself is 1. Neither was

willing to listen to an explanation of why both were wrong; they wanted only a verdict as to which was right. This would make a better story if there were a third who claimed that $0/0=\infty$ because anything divided by 0 is ∞, but I shall not yield to the temptation to embellish the facts.

Then there is the teacher who seriously claimed that it was literally correct to write $1/0=\infty$ in a trigonometry class on the grounds that Veblen and Young write this equation in their Projective Geometry. Needless to say, Veblen and Young are considering a particular coordinate system on a line in the projective plane and division means to them not numerical division, but a certain quadrilateral construction.

Another instructor "derived" an equation for vertical lines as follows:

$$y - b = \infty\,(x - a),$$

$$x - a = \frac{1}{\infty}(y - b) = 0.$$

In fact, so many people and textbooks have the courage of their confusion on the subject of infinity that it is small wonder that some of our brightest students come away with the impression that the cardinal number of the set of all natural numbers is ∞ which is obtained by dividing 1 by 0 and represents the place where parallel lines meet. I have often wondered why no one has ever "proved" the statement $1/0=\infty$ by appealing to the fact that if one tries to divide 1 by 0 on a calculating machine it will run until worn out.

Then there are many books which carelessly ask the student to write the nth term of a sequence whose first few terms are given, as though this were a reasonable mathematical question. To answer it, as you know, requires not mathematics but clairvoyance. For example, the nth term of a sequence beginning with 2, 4, 6, 8, $\cdots$ may be $2n$ but it may also be $2n+(n-1)\,(n-2)\,(n-3)\,(n-4)f(n)$ where $f(n)$ is almost completely arbitrary.

Perhaps more serious than these troublesome confusions is the failure of some teachers and some textbooks to pay due attention to converse propositions. For example, one of my ex-colleagues once remarked to me that he liked trigonometric identities because nothing could be more satisfyingly certain than to come out with the same expression on both sides. I answered, "Yes, as long as all the steps are reversible." The man's jaw dropped. He had never heard of that, and it took me 45 minutes to persuade him that the trouble lay with the converse proposition. It turned out that, although he liked trigonometric identities and had taught them for years, he had never understood them. Similarly, many texts on analytic geometry leave the student without the slightest suspicion that a converse proposition is needed in discussing equations of loci.

I say that this is perhaps a more serious matter because it is symptomatic of a more or less widespread tendency that I regard as vicious: namely, the

tendency to *say* that one of the main objectives of teaching mathematics is to train the student to think logically and then to *teach* the subject as though it had nothing to do with logical reasoning at all. The man just referred to paid careful attention to converses in plane geometry because that was traditional but had little suspicion that algebra or trigonometry had anything to do with reasoning. In fact, I recently had the distressing experience of hearing someone high in educational circles say that after all plane geometry is mostly a matter of memory. Is it any wonder that this opinion of mathematics is so widespread when it is all too often taught as a matter of memorized formulas, routine meaningless manipulations, and undigested proofs? It is my opinion that many of the ills of mathematics come from such excessive formalization and neglect of fundamental principles and common sense. If mathematics is presented as a collection of arbitrary rules of thumb for performing peculiar manipulations which somehow solve problems in which the student is not interested anyway, it is not surprising that many intelligent students are repelled from the subject for life. Such a collection of tricks is merely a branch of parlor magic, while the ideas of mathematics constitute an important current in the history of human thought. When these disgruntled individuals grow up some of them lead movements to abolish mathematics as a required subject. I am convinced that this often happens because they have never had contact with genuine mathematics at all, but only with formalized, memorized regurgitated techniques.

This souring of public opinion among non-technical people is by no means the only ill effect of excessive formalization. I am convinced that many purely technical difficulties arise from the same source. Thus, the errors given in (2) arise from thoughtless, superficial, formal analogy with the distributive law $a(x+y)=ax+ay$. The troubles with cancellation, transposition, and fractions ((1), (3), (4)) are due to thoughtless, unrationalized formalization. A memorized rule that has not been understood is difficult to recapture when the smallest part of it is forgotten.

I do not know whether one can attribute the tendency toward excessive formalization to a unique source. Perhaps it is partly due to the fact that teachers are sometimes judged by the results of their students on stereotyped examinations which can usually be passed with only formal techniques and without understanding. This practice may put pressure on the teacher to forget about the difficult (though worth while) task of teaching mathematics and cause him to give instead a coaching course for the examination. This makes the teacher's life easier, but harder to justify. He can then "cover" all the topics required for the examination easily, by the simple process of cramming memorized techniques into his students and ignoring the opportunity to give real insight into mathematics. He can teach "box" methods for solving problems since he is sure that the examination will not contain a strange "type" of problem that will not fit into his boxes. I have actually seen a secondary teacher who taught the logarithmic solution of triangles at the outset of the term before the very

definitions of the trigonometric functions. He "justified" this procedure on the grounds that he found students lost most credits on those questions on the Regents' examinations and consequently should get as much drill as possible on them. Beginning on the first day of the term surely gives the maximum drill. I met another secondary teacher who disliked teaching plane geometry because he was unable to formalize it as he did algebra. He did, however, like to teach some "calculus" in his advanced algebra section. By "teaching calculus" he seems to have meant training his students to write nx^{n-1} whenever they saw x^n, much as a mouse is trained to run a maze when a gong rings.

By now, I think I have complained enough about some of the extreme cases that I have had the misfortune to encounter. I hope they are uncommon. Let me now say a few words about some much less rare situations. There is the teacher who does "take up" the fundamental ideas and the reasoning behind his processes. But he does it once and rapidly, hopes that the best students will get it, and races on. This practice arises from a point of view with which I am not in sympathy. I do not consider the fundamentals important only for the few best students and the techniques important for everybody. I think that the fundamentals are important for all and will justify, in the long run, the time and effort needed to get them over to most of the class. I think it is part of the teacher's job to disseminate the ideas of mathematics as widely as possible, not to keep them within an esoteric circle of witch-doctors. The "devil take the hindmost" theory of teaching, if pushed to extremes, leads the teacher to give lectures which only he understands.

Another practice of which I disapprove and which is prevalent among college teachers, is that of blaming the student's incompetence on his high school training and letting it go at that. In fact, the high school teachers proceed similarly to blame the elementary school teachers, who blame the parents, who presumably can blame nothing but heredity, so that the original fault seems to lie with Adam. I can see no sense whatever in grumbling because a freshman has not mastered the routine techniques of high school algebra and then rushing him on into further ill-understood memorized techniques. We must accept our students as incontrovertible data; if they never learned to add fractions sensibly let us not consider it beneath our dignity to teach it to them. Even where they have had prerequisite material, they often have it no longer. Unless one is willing to face the facts as they are and do the best job possible under the circumstances, one might as well lapse into one's anecdotage and go about mumbling about the good old days.

I have said a great deal about what I think a good teacher of freshman mathematics should not do. It is harder to say what he should do since that depends on his own point of view, his temperament, training, and interests. However I shall undertake to enunciate a few generalities.

I think it is important for a teacher of elementary mathematics to be thoroughly familiar with the foundations of the subjects he teaches and to have as

broad a mathematical background as possible. A teacher who has only superficial knowledge himself, slips comfortably into the habit of regarding things as "easy" because he is "familiar" with them. On the other hand, genuine familiarity with the deep-lying difficulties underlying elementary mathematics will, if nothing else, prevent false emphases and tend to make one more tolerant towards the students' difficulties. This was brought home to me in the first high school class I ever taught when a student bisected an angle thus:

An angle had been defined as a figure consisting of two rays with a common vertex. But insufficient stress had been placed on a careful definition of what is meant by "bisecting" an angle. For example, one may be more inclined to sympathize with a student's troubles with minus signs if one realizes oneself that $-(-2)$ and $(-1)\ (-2)$ represent different situations. In any case, you will not teach them that "two minuses make a plus" because you can lay one minus sign vertically across the other thus: $+$. Many an advanced student, well trained in advanced techniques, remains unfamiliar with fundamental concepts because he has been expected to acquire familiarity with them by osmosis. This process does not always take place, perhaps because it proceeds only from the more dense to the less dense.

I think it is important that you try not to teach falsehoods that have to be corrected later, unless, as Forder remarks,* it be contended that an unsound proof has an educational value not possessed by a sound one. Needless to say, an unsound proof may well have educational value not possessed by a sound one provided it is used as an example of faulty reasoning. By this I do not mean at all that elementary teaching should be rigorous. I mean only that where gaps occur they should be pointed out, and that theorems which cannot be treated soundly at the student's level should be assumed or discussed informally rather than be given a bad proof.

Try to keep alive your interest in mathematics and its teaching. It is easier to give your students enthusiasm for mathematics if you have it yourself. Despite heavy schedules and other adverse conditions, it is always possible to come yet a little closer to attaining your ideal objectives, provided you know what you want them to be.† Try to teach in such a way that the students will acquire:

* H. G. Forder, The Foundations of Euclidean Geometry, Cambridge Press, 1927, p. viii.

† For an excellent discussion of the objectives of teaching elementary mathematics, see the Fifteenth Yearbook of the National Council of Teachers of Mathematics, which was drawn up by a joint committee of the Council and the Mathematical Association of America.

(1) an appreciation of the natural origin and evolutionary growth of the basic mathematical ideas from antiquity to the present; (2) a critical logical attitude and a wholesome respect for correct reasoning, precise definitions, and clear grasp of underlying assumptions; (3) an understanding of the role of mathematics as one of the major branches of human endeavor and its relations with other branches. Try to emphasize the distinction between familiarity and understanding, between proof and routine manipulation, between a critical attitude of mind and habitual unquestioning belief, between scientifically organized knowledge and both encyclopedic collections of facts and mere opinion and conjecture. Try to give them not only formulas but a wholesome appreciation of the nature and importance of mathematics.

Let me try to summarize my point of view in another way. There are three main aspects that must be brought out to some extent in the teaching of elementary mathematics, namely:

A. The routine techniques which are basic for future work. This aspect will be designated by "*techniques*."

B. The concrete applications of mathematics, the concrete settings in which mathematics originated, the interrelations among the "branches" of mathematics, and it relevance to the real world in general. For lack of a better term, this aspect will be referred to as the "*relevance*" of mathematics.

C. The reasonable justification of the techniques, and the logical nature of mathematics in general. This aspect will be designated by "*reasonableness*."

There seems to be some fear in certain quarters that overemphasis of C leads to excessive abstraction which will repel the general public from mathematics. This fear seems to be founded on the undeniable fact that some abstractions which appear in research papers lack a rich concrete background. This is true, but it is also true that few mathematicians are seriously interested in such an abstraction per se except possibly for the purpose of acquiring a Ph.D. or a promotion. Nevertheless, it must be granted that even such an empty abstract science is pure mathematics even though it is not important pure mathematics. It does not seem to me that anything is to be gained by hiding this innocent fact from the general public as though it were an important trade secret, much as the Pythagoreans are said to have tried to hide the irrationality of the square root of two. The truth will gain more for our subject than suppression of allegedly harmful ideas, which are, in my opinion, not harmful at all as long as one emphasizes that mathematicians are almost always interested only in abstractions which have rich concrete interpretations and which serve to unify or clarify concrete subjects which are interesting in themselves. As for this tendency to abstraction seeping harmfully into the teaching of elementary mathematics, such fear strikes me as being not only groundless but actually pointed in precisely the wrong direction.

In the actual practice of elementary teaching, as I have observed it, "techniques," "relevance," and "reasonablesness" are usually given attention in precisely that order, with "techniques" far in the lead almost to the total exclu-

sion of the other two, with "relevance" running a very bad second, and "reasonableness" a much worse third. It is my contention that this order should be reversed; in any case "reasonableness" and "relevance" deserve at least as much importance as "techniques." For students whose major interest is in the sciences, all three aspects should be virtually tied for first place. By this, I do not mean at all to belittle the importance of "techniques" for technical students, but I am convinced that when their reasonable justification is properly stressed, the techniques themselves require far less attention than is ordinarily given them. Understanding really does improve technical facility.

For students with little or no interest in the sciences, who really have little need for such facility, "techniques" should trail considerably behind. It is definitely my experience that these students can become very interested in the logical nature of mathematics, whereas they have no personal interest whatever in solving problems of a practical (to a technician) nature, no matter how interesting such problems may seem to a mathematician. It may seem surprising to many teachers, but it is nevertheless a fact that to many college freshmen the idea that algebra is a reasonable subject comes as a complete revelation. If there is danger to the status of mathematics, it does not arise from overemphasis of its "reasonableness." It comes from the deadly overemphasis on routine "techniques," and the unwholesome neglect of its "reasonableness" and of its "relevance" to the real world.

THE FIRST ANNUAL MEETING OF THE METROPOLITAN NEW YORK SECTION

The first annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Hunter College, New York City, on Saturday, April 18, 1942. Professor F. H. Miller presided at the morning session and Professor T. F. Cope, chairman of the Section, presided at the afternoon meeting.

The attendance was about one hundred and thirty-six, including the following seventy members of the Association: R. Lucile Anderson, R. G. Archibald, L. A. Aroian, Aaron Bakst, Brother Bernard Alfred (Welch), Frank Boehm, A. B. Brown, Jewell Hughes Bushey, J. H. Bushey, S. S. Cairns, H. R. Cooley, Elizabeth M. Cooper, T. F. Cope, W. H. H. Cowles, Marguerite D. Darkow, D. R. Davis, Carolyn Eisele, W. H. Fagerstrom, J. M. Feld, Edward Fleisher, R. M. Foster, B. P. Gill, Marion C. Gray, Etta Greenberg, Harriet M. Griffin, J. I. Griffin, C. C. Grove, N. A. Hall, L. S. Hill, J. H. Hlavaty, R. A. Johnson, L. S. Kennison, E. H. Koch, Jr., Edna E. Kramer-Lasser, Helen L. Kutman, A. W. Landers, Mary K. Landers, Nathan Lazar, C. H. Lehmann, C. C. MacDuffee, H. F. Mac Neish, J. J. McCarthy, P. H. McGrath, Mary McKenna, D. May Hickey Maria, A. E. Meder, F. H. Miller, E. C. Molina, Philip Newman, M. A. Nordgaard, Walter Penney, Mina S. Rees, Selby Robinson, S. G. Roth, H. D. Ruderman, Arthur Sard, Edna C. Schnefel, L. P. Siceloff, Lao G. Simons,

James Singer, C. S. Stuckey, J. A. Swenson, J. J. Tanzola, H. E. Wahlert, Louis Weisner, Mary E. Wells, A. Marie Whelan, D. E. Whitford, John Williamson, Jack Wolfe.

At the beginning of the morning session President G. N. Shuster of Hunter College welcomed the Section to Hunter College. At the beginning of the afternoon meeting the following officers were elected for the coming year: Chairman, H. F. Mac Neish, Brooklyn College; Vice-Chairman, Edna E. Kramer-Lasser, Thomas Jefferson High School; Secretary, H. E. Wahlert, New York University; Treasurer, F. H. Miller, Cooper Union.

The following four papers were presented at the morning session:

1. "An application of matrix theory to cryptography" by Professor L. S. Hill, Hunter College.

2. "The teaching of mathematics at the defense training institute" by C. H. Lehmann, Cooper Union.

3. "On the principles of statistical inference" by Dr. Abraham Wald, Columbia University and Queens College, introduced by Professor Cope.

4. "Mathematical training for aeronautical engineers" by Dr. N. A. Hall, Vought-Sikorsky Aircraft Company.

At the afternoon session a symposium on "Integrated mathematics in high school" was held. Dr. J. A. Swenson of Andrew Jackson High School presided at this symposium and the following four papers were presented:

5. "The significance of Δx in secondary mathematics" by Agnes Morley, Andrew Jackson High School, introduced by Dr. Swenson.

6. "Integrated mathematics with special application to the tenth year (geometry)" by Harry Sitomer, New Utrecht High School, Brooklyn, introduced by Dr. Swenson.

7. "Spatial and probable relationships in secondary mathematics" by Dr. Edna E. Kramer-Lasser, Thomas Jefferson High School, Brooklyn.

8. "Integrated mathematics in Catholic high schools" by Brother Anselm, St. Joseph's Normal Institute, introduced by Brother Bernard Alfred.

Abstracts of the papers follow.

1. Professor Hill's presentation was based upon two articles published in this MONTHLY: *Cryptography in an algebraic alphabet*, July, 1929, and *Certain linear transformation apparatus of cryptography*, March, 1931. The articles attracted wholly unanticipated attention in the United States and abroad. Their interest was strengthened by the circumstance that the author's learned, ingenious, and mechanically-minded colleague of some fifteen years, Professor Louis Weisner, was able to plan a machine for the speedy and effortless operation of those simpler types of transformation which were considered in the 1929 article. These transformations include, as special cases several of the more prominent military cipher systems, and afford extensions which seem to be quite beyond cryptanalytic approach.

2. Mr. Lehmann described the origin, organization, and scope of the Defense Training Institute of the Engineering Colleges of Greater New York. This new

school, in actual operation since February 10, 1941, is the only one in the country having a full daytime coordinated curriculum set up primarily to train engineering personnel for war industries. The address featured the place of mathematics in the Institute's curriculum and described some of the teaching problems in mathematics encountered in what is probably one of the most highly concentrated and intensive technical programs ever undertaken.

3. Dr. Wald discussed some modern developments of the theory of statistical estimation. Suppose x is a random variable and its probability distribution function $f(x, \theta)$ involves an unknown parameter θ. In the theory of estimation the value of the unknown parameter θ has to be estimated on the basis of N observations $x_1, \cdots, x_N$ on the variable x, *i.e.*, we have to construct a function $t(x_1, \cdots, x_N)$ of the observations which can be considered as a "good statistical estimate" of the parameter θ. In the modern theory definitions for a "good" or "best" statistical estimate are formulated and solutions are given for certain classes of cases. If the number of observations N is large, the so-called maximum likelihood estimate provides a satisfactory solution of the problem of estimation under fairly general conditions.

4. Dr. Hall discussed the mathematical training required by aeronautical engineers from the viewpoint of the engineer, whose primary consideration is the production of the most efficient airplane. He presented a brief picture of the program of development of airplane design from the original conception in terms of function and specifications to the final detail design for production. The various places where mathematics entered the program were pointed out to indicate the type of mathematical training required. It was observed that two distinct types of ability were demanded. First, the average engineer was called upon to use mathematics of a difficulty up to the grade of the calculus with absolute accuracy and dependability. The need for training in thoroughly reliable work as a basic requirement was stressed. The second demand was for a preparation in advanced fields of mathematical analysis with emphasis on numerical methods and the possession of the viewpoint of mathematical service to the engineering program. "When we are training aeronautical engineers, we must train men to develop airplanes and not mathematics."

6. Mr. Sitomer pointed out that, whether in the tenth year or not, integrated mathematics must not be confused with simultaneous, correlated, comprehensive, or general mathematics, which refer to the scope of instructional materials but not to the methods of presentation and solution. Integrated mathematics may be defined as a program, not a syllabus, which aims to enlist any branch of mathematics which is best fitted to solve a given problem, whether it be computation or proof. The more branches in use, the better the integration. The simpler the computation or proof, the better the integration. In the tenth year we may select methods from algebra, plane and solid geometry, trigonometry, calculus, function theory, number theory, vector analysis and topology. Courageous experimentation in the classroom by teachers, well-read in mathematics, will determine possible syllabi. In the tenth year special emphasis on the

nature of proof and on a postulational system demands a simple synthetic introduction with a gradual transition to analytic geometry and other methods of proofs. Integrated mathematics results in a rich set of concepts arranged spirally in a syllabus.

7. Dr. Kramer-Lasser showed how integrated mathematics is used to develop general space concepts. Even as early as the seventh year, elementary procedures form a foundation for more advanced work. For example, in working with paper patterns, proper numbering of vertices and edges eventually leads pupils to think of the surfaces they construct as sets of triangles with suitable identification of vertices and edges. In the ninth and tenth years, space geometry is studied along with plane. By rotating and translating 3, 4, $\cdots$, $n-1$ dimensional forms, pupils obtain an abstract realization of 4, 5, $\cdots$, n dimensional figures. Combinatorial methods are also used to obtain the concept of higher dimensions. In the eleventh and twelfth years, elementary analytic geometry of three dimensions is studied and analytic generalization to higher dimensions is made.

8. Brother Anselm stated that, in general, the Catholic High Schools in and around New York City adhere to the traditional courses in mathematics. They do this because they think that their present curriculum is the best for their particular needs. They are reluctant to change to the integrated course because most of the leading educators in this field are not certain that integrated mathematics courses are better than the traditional courses. Moreover, they fear that lack of drill and understanding of the reasons for manipulative processes may be brought about by the integrated program. They are convinced also that the much desired purpose of an integrated program can be achieved with the traditional subjects. Hence the Catholic schools look, not to a change of courses to improve secondary mathematics, but rather to better teaching of the present courses.

H. E. WAHLERT, *Secretary*

THE ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-sixth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado School of Mines, Golden, Colorado, April 17 and 18, 1942. There were three sessions. Professor J. C. Fitterer presided at the first two and at the business session of the third. The Saturday morning session was a joint meeting with the mathematics section of the Eastern Division of the Colorado Education Association. Mr. H. W. Charlesworth of East Denver High School presided at this meeting.

The attendance was forty-four, including the following fifteen members of the Association: C. F. Barr, J. R. Everett, J. C. Fitterer, I. L. Hebel, A. J. Kempner, Claribel Kendall, A. J. Lewis, S. L. Macdonald, A. E. Mallory, W. K.

Nelson, Greta Neubauer, M. G. Pawley, A. W. Recht, C. H. Sisam, and W. E. Wilson.

At the business meeting the following officers were elected for the coming year: Chairman, Professor A. E. Mallory, Colorado State College of Education; Vice-Chairman, Professor A. W. Recht, University of Denver.

The following papers were presented:

1. "An improved cosine-law slide rule" by Professor I. L. Hebel, Colorado School of Mines.

2. "On determining the 'best' critical region for testing the null hypothesis when the parent populations follow the Poisson law" by H. T. Guard, Colorado State College, introduced by Professor Macdonald.

3. "The use of Cauchy's integral formula in evaluating certain improper integrals" by Professor A. J. Lewis, University of Denver.

4. "A study of roulettes with the aid of the cathode-ray oscillograph" by Professor M. G. Pawley, Colorado School of Mines.

5. "Geometrical demonstration of a theorem on envelopes and its application to solve a maximum and minimum problem" by G. E. Uhrich, University of Colorado, introduced by Professor Kempner.

6. "The mathematical approach to the fundamentals of hydraulics" by C. P. Vetter, Senior Engineer, Bureau of Reclamation, introduced by the Secretary.

7. "Remarks on repeating decimal fractions" by Professor Emeritus I. M. De Long, University of Colorado. Read by Professor A. J. Kempner.

8. "Periodic decimal fractions, primitive roots and quadratic residues" by Professor A. J. Kempner, University of Colorado.

9. "A report of instruction and learning activities as observed in geometry class rooms" by Professor A. E. Mallory, Colorado State College of Education.

10. "The handmaiden spurned" by Professor O. H. Rechard and Professor C. F. Barr, the University of Wyoming.

11. "Some simple proofs of the addition law in trigonometry" by G. E. Uhrich, University of Colorado, introduced by Professor Kempner.

12. "The use of mathematics in industry" by L. A. McElroy, Denver public Schools, introduced by the Secretary.

13. "Constructions by means of a marked ruler and other instruments" by Professor Claribel Kendall, University of Colorado.

Abstracts of the papers follow.

1. Professor Hebel gave an extension of an earlier paper concerning a slide rule for the solution of the cosine law of spherical trigonometry, particularly as applied to distances on the earth. By properly manipulating the equation, an improved slide rule has been devised which gives a direct solution by purely mechanical means, as contrasted to the original model where it was necessary to "take out" intermediate answers to reach the final results. The operation was demonstrated on a specially constructed two meter long classroom model slide rule.

2. This problem was first considered by Przyborowski and Wilenski in *Biometrica*, 1939.

Given the random variables x_i where $\rho(x_i|\lambda_i)=\lambda_i^{x_i}e^{-x_i}/x_i!$, $(i=1, 2)$. To determine the best critical region for testing the composite hypothesis, H_0, that $\lambda_1=\lambda_2$ with respect to the alternative hypothesis $\lambda_1<\lambda_2$. The test consists in finding a region, w_0, in the sample space, W, such that, if E is the observed sample point,

1. $P\{E\epsilon w_0 | H_0\} \leqq \epsilon$, the desired level of significance.
2. $P\{E\epsilon w_0 | H_1\} > P\{E\epsilon w | H_1\}$

where H_1 is any alternative hypothesis and w is any other region that satifies 1.

The test reduces to rejection of H_0 when $x_1 \leqq x_\epsilon$ where x_ϵ is chosen so that

$$\sum_{x_1=0}^{x_\epsilon} \binom{s}{x_1} \left(\frac{1}{1+A}\right)^{x_1} \left(\frac{A}{1+A}\right)^{s-x_1} \leqq \epsilon$$

is satisfied. A is the ratio of the sample sizes and $s=x_1+x_2$.

The power function is

$$B(\theta | s) = \sum_{x_1=0}^{x_\epsilon} \binom{s}{x_1} \left(\frac{1}{1+A\theta}\right)^{x_1} \left(\frac{A\theta}{1+A\theta}\right)^{s-x} \quad \text{where} \quad \theta > 1.$$

The calculations of tables given the critical region and values of the power function for specified values of s and θ and also with s unspecified were demonstrated by Mr. Guard.

3. Professor Lewis outlined methods of using Cauchy's integral formula and Cauchy's residue formula in evaluation of certain improper integrals with real integrands. He illustrated the method by applying it to two examples.

4. Professor Pawley made a study of the parametric equation

$$x = a_1 \cos n_1 t + a_2 \cos n_2 t + a_3 \cos n_3 t,$$

$$y = a_1 \sin n_1 t \pm a_2 \sin n_2 t + a_3 \sin n_3 t.$$

The equation was shown to represent a group of roulettes, including epicycloids, hypocycloids, epitrochoids, and hypotrochoids, depending upon the relations between the a's and n's. The various curves were pictured on the flourescent screen of a cathode-ray oscillograph. Simple rules were given for anticipating the form of roulette obtained when the a's and n's in the equation are varied. An equation was given for a curve approximating the N-sided regular polygon and these curves were displayed on the screen of the oscillograph.

5. Mr. Uhrich dealt mainly with a geometrical proof of the theorem: The envelope of the family of chords which cut segments of equal area from a given curve is tangent to each chord at its midpoint. The converse of this theorem is proved in Salmon's *Analytische Geometrie der Kegelschnitte*, Kap. 13, with the purpose of application to conic sections; however the method seems to be general.

Besides some special examples, Mr. Uhrich used this theorem to prove: Of all the chords which can be drawn through a fixed point within a closed convex curve, that chord which is bisected by the point cuts off a segment of minimum area from the curve. This theorem was given as a problem in the *Analyst*, vol. 5.

6. Mr. Vetter outlined the various methods of mathematical approach to the solution of problems connected with fluid flow. These methods permit of practical solutions and of solutions that may be verified experimentally only in comparatively few and comparatively simple cases. He demonstrated further, that as an alternative it is possible to attack the problems from purely dimensional considerations without the necessity of simplifying assumptions. The latter method, without leading to final answers, nevertheless gives precise information as to the mathematical form of the functional relationship between the physical quantities involved and, in many instances, permits experimental verification or experimental evaluation of specific functional constants. He analyzed the relationship of the method to the theory of models.

7. Professor Emeritus Ira M. De Long of the University of Colorado is eighty-seven years old, and has been for four months in a Boulder hospital. He has maintained his life-long interest in the properties of repeating decimals, and never tires of telling me interesting and amusing relations which he has discovered for himself. Without claims of priority I present two examples.

(1) If we expand a fraction a/p, p a prime, say $3/7$, we have $10\cdot 3=4\cdot 7+2$, $10\cdot 2=2\cdot 7+6$, $10\cdot 6=8\cdot 7+4$, $10\cdot 4=5\cdot 7+5$, $10\cdot 5=7\cdot 7+1$, $10\cdot 1=1\cdot 7+3$, with the digit period $428571'$ and the remainder period 264513. The $7(4+2+8+5+7+1)=9(2+6+4+5+1+3)$. For a/p, $p\cdot\sum(\text{digits})=9\cdot\sum(\text{remainder})$. For a base g instead of 10, $g\neq 1$, $\not\equiv 0 \bmod p$, 9 is replaced by $g-1$.

(2) To obtain the period of $1/49$ from the period 142857 or $1/7$, proceed as follows: Write

.020408	020408	020408	020408	020408	020408	020408
	142857	285714	428571	571428	714285	857142
020408	163265	306122	448979	591836	734693	877550

which is (except for the last 0 which has to be replaced by 1) the correct period.

8. The examination of the period length and digit properties, etc., of an ordinary repeating decimal have a particularly special character, on account of the orientation with respect to the base 10. By considering simultaneously the representation of all fractions a/p, $(a/p)=1$, for a given p, with respect to *all* bases $g>1$, and by fixing attention on the periodic ***remainder*** sequences rather than upon the periodic *digit* sequences, the general pattern or relations appears clearly. For a/p, $a=1, 2, \cdots, p-1$ we have for each base $g>1$, $(p/g)=1$, the following theorems:

I. Let $d_1=1, \cdots, d_\nu, \cdots, d_\mu=p-1$ be the divisors of $p-1$. Then, denoting

by $L(a/p)_g$ the period length in the remainder sequence, there are exactly $\phi(d_r)$ remainder sequences for which $L(a/p)_g = d_r$. (For $p=7$, $a=4$, $g=2, 3, \cdots, 6, 8$ we have remainder sequences 124, 513264, 214, 623154, 34, 4, respectively.) The same holds of course for the digit periods.

II. For a given g the remainders $1, 2, \cdots, p-1$, of a given a/p will either form exactly a remainder period, or the $p-1$ remainders break up into α sets of β each, $\alpha\beta = p-1$. In each of these $L(a/p)_g$ one of the three cases occur: Either each number of the period is quadratic residue of p, or each number is a non-residue; or the numbers are alternately residues or non-residues.

III. If $(g/p)=1$, $L(a/p)_g$, is a divisor of $(p-1)/2$, 1 and $(p-1)/2$ inclusive. But if $(g/p)=-1$, $L(a/p)_g$, which must divide $p-1$, does not divide $(p-1)/2$.

9. Professor Mallory gave a report of the type of work being done in certain geometry classes. The classes visited showed a high degree of understanding of the subject matter as revealed by the procedures.

10. The paper by Professor Rechard and Professor Barr grew out of the rejection by a geologist of a simple formula involving a few trigonometric functions on the basis that geologists would not use it. Instead, in his published paper, he described and illustrated a graphic method for solving a chain of right triangles. Where the fault lies for such a situation is the question raised by this paper, in the hope that out of it might come some clue to the old problem of how to make usable mathematics used.

11. In this paper Mr. Uhrich gave a presentation of three of the four proofs with slight variations for the addition theorem for the sine of the sum or difference of two angles which are given in Professor Gerhard Hessenberg's *Ebene und sphärische Trigonometrie*, Kap. IV.

12. The successful person in industry is the one who can separate essential and non-essential factors, get the job done and show a profit. Mathematical training is important to the extent that it applies to the job in hand. Pupils should be offered basic knowledge of mathematics with the problems that are *real to them*. This should help them to develop the habit of applying mathematical knowledge to practical situations. Industry specializes; mathematics essential to one industry has no value to another. But, since no one can predict with 100% accuracy the potentialities of an individual, it would seem wise to recommend training in mathematics for each pupil, at least until a choice of occupation has been made.

13. We are all familiar with constructions by means of an unmarked ruler and compasses, the euclidean instruments. Some or all of these constructions can be carried out by means of other instruments such as an angle ruler, a marked ruler, an unmarked ruler without compasses or by compasses alone. Professor Kendall paid particular attention to constructions by means of a marked ruler, including the trisection of the angle and the finding of a length which is the cube root of a given line segment.

A. J. LEWIS, *Secretary*

MODIFIED CONTINUED FRACTIONS

J. W. BRADSHAW, University of Michigan

In an article* in this MONTHLY the "modification" of slowly convergent continued fractions was suggested by the author as a means of obtaining forms that would converge more rapidly and thus be more useful in computation. In the present article the general considerations underlying such "modification" are developed and the procedure is illustrated with some concrete examples. The notion of a modified continued fraction is not new; Sylvester† in 1869 called attention to a first modification of Euler's continued fraction for $\pi/2$, and Glaisher‡ in 1874 compared this result with a similarly modified Wallis product. But its usefulness seems not to have been appreciated.

1. Definition. Given the convergent infinite continued fraction $b_0+a_1/b_1+a_2/b_2+\cdots$, by $K_\nu=A_\nu/B_\nu$ we denote the convergent of order ν, and by $L_\nu=C_\nu/D_\nu$, the same finite continued fraction with the last partial quotient, a_ν/b_ν, replaced by c_ν/d_ν. The problem is to determine c_ν/d_ν so that

(a) $\lim_{\nu\to\infty} L_\nu=\lim_{\nu\to\infty} K_\nu$,

(b) L_ν will converge more rapidly than K_ν.

We call the function of ν thus obtained the convergent of order ν of the "modified continued fraction."

2. Formulation of conditions. The variation with ν of the convergents of the continued fraction and the modified continued fraction is governed by the two pairs of equations:

$$(1)\qquad \begin{cases} A_\nu = b_\nu A_{\nu-1} + a_\nu A_{\nu-2}, \\ B_\nu = b_\nu B_{\nu-1} + a_\nu B_{\nu-2}, \end{cases} \qquad (2)\qquad \begin{cases} C_\nu = d_\nu A_{\nu-1} + c_\nu A_{\nu-2}, \\ D_\nu = d_\nu B_{\nu-1} + c_\nu B_{\nu-2}, \end{cases}$$

$(\nu = 1, 2, 3, \cdots$; it being assumed as usual that $A_{-1} = 1$, $B_{-1} = 0)$.

From the relation

$$(3)\qquad A_\nu B_{\nu-1} - A_{\nu-1}B_\nu = (-1)^{\nu-1}a_1a_2\cdots a_\nu, \quad (\nu = 1, 2, 3, \cdots),$$

it follows that

$$(4)\qquad A_\nu/B_\nu - C_\nu/D_\nu = (-1)^{\nu-2}a_1a_2\cdots a_{\nu-1}(b_\nu c_\nu - a_\nu d_\nu) \div [b_\nu d_\nu B_{\nu-1}^2 + (b_\nu c_\nu + a_\nu d_\nu)B_{\nu-1}B_{\nu-2} + a_\nu c_\nu B_{\nu-2}^2], \quad (\nu = 2, 3, \cdots).$$

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* J. W. Bradshaw, Modified series, this MONTHLY, vol. 46, 1939, p. 486.

† J. J. Sylvester, Note on a new continued fraction applicable to the quadrature of the circle, Philosophical Magazine, Series 4, vol. 37, 1869, pp. 373–375.

‡ J. W. L. Glaisher, On the transformation of continued products into continued fractions, Proceedings of the London Mathematical Society, vol. 5, 1874, pp. 78–88.

If L_ν is to approach the same limit as K_ν, the limit of this expression must be zero. Since the continued fraction is assumed to be convergent,

$$\lim_{\nu\to\infty}(A_\nu/B_\nu - A_{\nu-1}/B_{\nu-1}) = \lim_{\nu\to\infty}(-1)^{\nu-1}a_1a_2\cdots a_\nu/B_\nu B_{\nu-1}$$
$$= \lim_{\nu\to\infty}(-1)^{\nu-1}a_1a_2\cdots a_\nu/(b_\nu B_{\nu-1}^2 + a_\nu B_{\nu-1}B_{\nu-2}) = 0.$$

The condition $\lim_{\nu\to\infty} L_\nu = \lim_{\nu\to\infty} K_\nu$ will then certainly be satisfied, if

$$(5)\qquad \begin{aligned} &|a_\nu d_\nu - b_\nu c_\nu| \leqq |a_\nu d_\nu|, \\ &|b_\nu d_\nu B_{\nu-1}^2 + (b_\nu c_\nu + a_\nu d_\nu)B_{\nu-1}B_{\nu-2} + a_\nu c_\nu B_{\nu-2}^2| \qquad (\nu = 2, 3, \cdots) \\ &\qquad \geqq |b_\nu d_\nu B_{\nu-1}^2 + a_\nu d_\nu B_{\nu-1}B_{\nu-2}|. \end{aligned}$$

Now further, we wish $C_\nu/D_\nu - C_{\nu-1}/D_{\nu-1}$ to approach zero faster than $A_\nu/B_\nu - A_{\nu-1}/B_{\nu-1}$. Use of the reduction formulas (1) and (2) yields

$$C_\nu D_{\nu-1} - C_{\nu-1}D_\nu = (-1)^{\nu-2}a_1a_2\cdots a_{\nu-2}(d_\nu d_{\nu-1}a_{\nu-1} - d_\nu c_{\nu-1}b_{\nu-1} - c_\nu c_{\nu-1}),$$
$$(\nu = 2, 3, \cdots, \text{ if for } \nu = 2 \text{ we put } a_1a_2\cdots a_{\nu-2} = 1).$$

The comparative rate of approach to zero is then given by

$$(6)\qquad \begin{aligned} &(C_\nu/D_\nu - C_{\nu-1}/D_{\nu-1})/(A_\nu/B_\nu - A_{\nu-1}/B_{\nu-1}) \\ &\quad = -B_\nu B_{\nu-1}(d_\nu d_{\nu-1}a_{\nu-1} - d_\nu c_{\nu-1}b_{\nu-1} - c_\nu c_{\nu-1})/D_\nu D_{\nu-1}a_\nu a_{\nu-1}, \quad (\nu = 2, 3, \cdots). \end{aligned}$$

The limit of this multiplied by ν^α should equal a finite quantity G for some positive α.

3. Application to a continued fraction for $\pi/2$. As an illustration we consider the continued fraction, $1+K_{\nu=1}^{\infty}\, 1/(1/\nu)$, which is equivalent to Euler's continued fraction for $\pi/2$. Here $b_0=1$, $a_\nu=1$, $b_\nu=1/\nu$; we assume $c_\nu=1$, and conditions (5) and (6) take the simpler form

$$(7)\qquad d_\nu > 0, \quad |d_\nu B_{\nu-1}^2 + (1+\nu d_\nu)B_{\nu-1}B_{\nu-2} + \nu B_{\nu-2}^2| \geqq |d_\nu B_{\nu-1}^2 + \nu d_\nu B_{\nu-1}B_{\nu-2}|,$$

$$(8)\qquad -B_\nu B_{\nu-1}(d_\nu d_{\nu-1} - d_\nu b_{\nu-1} - 1)\nu^\alpha/D_\nu D_{\nu-1} = G \neq 0 \text{ for a positive } \alpha.$$

For d_ν we choose a rational fraction in ν, seeking to determine coefficients so that

$$(9)\qquad d_\nu(b_{\nu-1} - d_{\nu-1}) + 1$$

shall be of low degree in ν. It is clear that numerator and denominator of d_ν should be of the same degree and have the same leading coefficient. Assume

$$(10)\qquad d_\nu = (q_0\nu^t + q_1\nu^{t-1} + \cdots + q_t)/(p_0\nu^t + p_1\nu^{t-1} + \cdots + p_t),$$
$$q_0 = p_0, \ t \text{ an integer.}$$

Then the numerator of the expression (9) takes the form

$$
\begin{aligned}
&(q_0\nu^t + q_1\nu^{t-1} + \cdots + q_t)\\
(11)\qquad &[-q_0(\nu-1)^{t+1} + (p_0 - q_1)(\nu-1)^t + \cdots + (p_{t-1} - q_t)(\nu-1) + p_t]\\
&+ (p_0\nu^t + p_1\nu^{t-1} + \cdots + p_t)[p_0(\nu-1)^{t+1} + p_1(\nu-1)^t + \cdots + p_t(\nu-1)],
\end{aligned}
$$

while the denominator is

$$
(12)\qquad (p_0\nu^t + p_1\nu^{t-1} + \cdots + p_t)[p_0(\nu-1)^{t+1} + p_1(\nu-1)^t + \cdots + p_t(\nu-1)].
$$

The number of terms, powers of ν including the zeroth, in (11) is $2t+2$, while the number of independent coefficients at our disposal is $2t+1$. It is therefore possible to make this numerator independent of ν. Equating to zero coefficients of successive powers of ν we obtain the sequence of equations:

$$
\begin{aligned}
&p_0 - q_0 = 0,\\
&p_1 - q_1 = -\frac{1}{2}p_0,\\
&p_2 - q_2 = -\frac{1}{2}p_1 - \frac{3}{8}p_0,\\
&p_3 - q_3 = -\frac{1}{2}p_2 - \frac{3}{8}p_1,\\
(13)\qquad &p_4 - q_4 = -\frac{1}{2}p_3 - \frac{3}{8}p_2 + \frac{45}{128}p_0,\\
&p_5 - q_5 = -\frac{1}{2}p_4 - \frac{3}{8}p_3 + \frac{45}{128}p_1,\\
&p_6 - q_6 = -\frac{1}{2}p_5 - \frac{3}{8}p_4 + \frac{45}{128}p_2 - \frac{1125}{1024}p_0,\\
&p_7 - q_7 = -\frac{1}{2}p_6 - \frac{3}{8}p_5 + \frac{45}{128}p_3 - \frac{1125}{1024}p_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}
$$

These yield the following sequence of possibilities for d_ν:

$$
\begin{aligned}
&\frac{4\nu-1}{4\nu-3} = 1 + \frac{2}{|4\nu-3},\\
(14)\qquad &\frac{16\nu^2-12\nu+11}{16\nu^2-20\nu+15} = 1 + \frac{2}{|4\nu-3} + \frac{9}{|4\nu-2},\\
&\frac{64\nu^3-80\nu^2+168\nu-47}{64\nu^3-112\nu^2+200\nu-105} = 1 + \frac{2}{|4\nu-3} + \frac{9}{|4\nu-2} + \frac{25}{|4\nu-2}.
\end{aligned}
$$

This suggests that the successive convergents of a certain infinite continued fraction will furnish the modification we desire.

4. Continued fraction form of modification. We now define a new continued fraction, using ρ as index and accents on the quantities connected with this to distinguish them from those of the continued fraction with which we started:

$$4\nu - 3 + \mathop{K}_{\rho=1}^{\infty} \frac{(2\rho+1)^2}{4\nu-2}. \tag{15}$$

The convergents of order 0, 1, and 2 have entered into the values of d_ν given above. We now seek to show that if $A'_\rho(\nu)/B'_\rho(\nu)$ denotes the convergent of order ρ of this fraction,

$$d_\nu = 1 + 2 \div A'_\rho(\nu)/B'_\rho(\nu) = [A'_\rho(\nu) + 2B'_\rho(\nu)]/A'_\rho(\nu) \tag{16}$$

furnishes the desired modification for $\rho = 0, 1, 2, \cdots$. The expression (9) now takes the form

$$\frac{A'_\rho(\nu) + 2B'_\rho(\nu)}{A'_\rho(\nu)}\left[\frac{1}{\nu-1} - \frac{A'_\rho(\nu-1) + 2B'_\rho(\nu-1)}{A'_\rho(\nu-1)}\right] + 1$$

$$= \frac{1}{(\nu-1)A'_\rho(\nu)A'_\rho(\nu-1)}\,[A'_\rho(\nu)A'_\rho(\nu-1) - 2(\nu-1)A'_\rho(\nu)B'_\rho(\nu-1)$$

$$- 2(\nu-2)A'_\rho(\nu-1)B'_\rho(\nu) - 4(\nu-1)B'_\rho(\nu)B'_\rho(\nu-1)].$$

The numerator of this fraction we denote by $M_\rho(\nu)$ and proceed to show that it is independent of ν. We have, in fact, $M_0(\nu) = -9$, $M_1(\nu) = 225$, $M_2(\nu) = -11025$, $M_3(\nu) = 893025$. By means of the reduction formulas $M_\rho(\nu)$ can be expressed in terms of $M_{\rho-1}(\nu)$, $M_{\rho-2}(\nu)$, and two new functions that we denote by $U_\rho(\nu)$ and $V_\rho(\nu)$:

$$U_\rho(\nu) = A'_\rho(\nu)A'_{\rho-1}(\nu-1) - 2(\nu-1)A'_\rho(\nu)B'_{\rho-1}(\nu-1)$$

$$- 2(\nu-2)B'_\rho(\nu)A'_{\rho-1}(\nu-1) - 4(\nu-1)B'_\rho(\nu)B'_{\rho-1}(\nu-1),$$

$$V_\rho(\nu) = A'_\rho(\nu-1)A'_{\rho-1}(\nu) - 2(\nu-1)B'_\rho(\nu-1)A'_{\rho-1}(\nu)$$

$$- 2(\nu-2)A'_\rho(\nu-1)B'_{\rho-1}(\nu) - 4(\nu-1)B'_\rho(\nu-1)B'_{\rho-1}(\nu).$$

We then have

$$M_\rho(\nu) = (4\nu-2)(4\nu-6)M_{\rho-1}(\nu) + (4\nu-2)(2\rho+1)^2U_{\rho-1}(\nu)$$

$$+ (4\nu-6)(2\rho+1)^2V_{\rho-1}(\nu) + (2\rho+1)^4M_{\rho-2}(\nu).$$

Further application of the reduction formulas yields

$$U_{\rho-1}(\nu) = (4\nu-2)M_{\rho-2}(\nu) + (2\rho-1)^2V_{\rho-2}(\nu),$$

$$V_{\rho-1}(\nu) = (4\nu-6)M_{\rho-2}(\nu) + (2\rho-1)^2U_{\rho-2}(\nu),$$

and then from $M_\rho(\nu)$ and $M_{\rho-2}(\nu)$ the quantities $U_{\rho-3}(\nu)$ and $V_{\rho-3}(\nu)$ can be eliminated, giving finally

$$
\begin{aligned}
M_\rho(\nu) = {} & (4\nu - 2)(4\nu - 6)M_{\rho-1}(\nu) \\
& + (2\rho + 1)^2(32\nu^2 - 64\nu + 40 + 8\rho^2 + 2)M_{\rho-2}(\nu) \\
& + (2\rho + 1)^2(2\rho - 1)^2(4\nu - 2)(4\nu - 6)M_{\rho-3}(\nu) \\
& - (2\rho + 1)^2(2\rho - 1)^2(2\rho - 3)^4 M_{\rho-4}(\nu).
\end{aligned} \tag{17}
$$

By mathematical induction it can then be shown that

$$
M_\rho(\nu) = (-1)^{\rho-1}3^2 \cdot 5^2 \cdot \cdots \cdot (2\rho + 3)^2, \quad (\rho = 0, 1, 2, \cdots). \tag{18}
$$

The original continued fraction converges very slowly; the value of $\pi/2$ obtained by its use shows an error of 15 in the second decimal place when ν is taken equal to 4, and even when $\nu = 8$, the error is still more than 8 in this second decimal place. This is to be contrasted with an error of 3 in the seventh decimal place obtained by using the modified continued fraction with $\nu = 4$, $\rho = 4$.

5. Other solutions. Other solutions of the problem proposed can be obtained by imposing different conditions on d_ν. If instead of requiring that (11) shall be independent of ν, we now require that it shall be the product of $\nu - 1$ and a quantity independent of ν, we obtain a second solution. This will enable us to cancel the factor $\nu - 1$ from numerator and denominator of d_ν. To this end we put $p_t = 0$ and proceed to determine coefficients as before. The sequence of values of d_ν are now convergents of a modified continued fraction, namely, the same continued fraction as before, except that the ρth partial quotient $(2\rho+1)^2/(4\nu-2)$ is to be replaced by $(2\rho+1)^2/(4\nu+2\rho+1)$. A suggested notation for this modified continued fraction is the following:

$$
4\nu - 3 + \overset{\rho}{\underset{1}{K}} \frac{(2\rho + 1)^2}{4\nu - 2} (m) \frac{(2\rho + 1)^2}{4\nu + 2\rho + 1}. \tag{19}
$$

The rate of convergence is somewhat improved as soon as $\rho \geqq 2\nu$.

A third solution may be obtained by arranging for further cancellation. Since we have taken $p_t = 0$, the denominator (12) is now divisible by $(\nu - 1)^2$. The numerator will also contain this factor and it may be cancelled, if we take $p_{t-1} - q_t = 0$. If the remaining coefficients are then determined, the result is the same continued fraction modified in a different way; the ρth partial quotient is $(2\rho+1)^2/(2\nu+2\rho+1)$:

$$
4\nu - 3 + \overset{\rho}{\underset{1}{K}} \frac{(2\rho + 1)^2}{4\nu - 2} (m) \frac{(2\rho + 1)^2}{2\nu + 2\rho + 1}. \tag{20}
$$

This modified continued fraction converges still more rapidly, if $\rho \geqq 2\nu$.

6. Double modification. The two modified continued fractions met above suggest the application of the general method directly to the continued fraction $4\nu - 3 + K_{\rho=1}^{\infty}(2\rho+1)^2/(4\nu-2)$. For this purpose we take $c'_\rho = a'_\rho = (2\rho+1)^2$ and

$$
d'_\rho = (q'_0\rho^{t+1} + q'_1\rho^t + \cdots + q'_{t+1})/(p'_0\rho^t + p'_1\rho^{t-1} + \cdots + p'_t).
$$

Condition (6) now leads us to try to make $d'_\rho(d'_{\rho-1}-b'_{\rho-1})-c'_\rho$ of low degree in ρ. The coefficients of powers of ρ in the numerator of this expression we equate to zero, obtaining the sequence of equations:

$$
\begin{aligned}
q'_0 - 2p'_0 &= 0,\\
q'_1 - 2p'_1 &= (2\nu+1)p'_0,\\
q'_2 - 2p'_2 &= (2\nu+1)p'_1 + \nu^2 p'_0,\\
q'_3 - 2p'_3 &= (2\nu+1)p'_2 + \nu^2 p'_1 - \nu^2 p'_0,\\
q'_4 - 2p'_4 &= (2\nu+1)p'_3 + \nu^2 p'_2 - \nu^2 p'_1 - \tfrac{1}{4}(\nu^4+2\nu^3-3\nu^2)p'_0,\\
q'_5 - 2p'_5 &= (2\nu+1)p'_4 + \nu^2 p'_3 - \nu^2 p'_2 - \tfrac{1}{4}(\nu^4+2\nu^3-3\nu^2)p'_1\\
&\quad + \tfrac{1}{4}(3\nu^4+6\nu^3-\nu^2)p'_0,\\
&\cdots\cdots\cdots

$$\rho: \quad A'_0 = 4\nu - 3, \qquad B'_0 = 1;$$
$$A'_\rho = (4\nu - 2)A'_{\rho-1} + (2\rho + 1)^2 A'_{\rho-2},$$
$$B'_\rho = (4\nu - 2)B'_{\rho-1} + (2\rho + 1)^2 B'_{\rho-2}, \qquad (\rho = 1, 2, 3, \cdots).$$
$$\sigma: \quad A''_0 = \tfrac{1}{4}(2\nu + 2\rho + 1), \qquad B''_0 = 1;$$
$$A''_\sigma = (\rho + 1)A''_{\sigma-1} + \tfrac{1}{4}(\nu - 1 + \sigma)^2 A''_{\sigma-2},$$
$$B''_\sigma = (\rho + 1)B''_{\sigma-1} + \tfrac{1}{4}(\nu - 1 + \sigma)^2 B''_{\sigma-2}, \qquad (\sigma = 1, 2, 3, \cdots).$$

By substituting in this formula we find, for example, that $f(4, 4, 4)$ yields an approximation with an error of 2 in the ninth decimal place. But we also note that $f(5, 4, 3)$ and $f(8, 4, 0)$ have exactly the same value. Investigation of a considerable number of such cases leads to the surmise that, in general,

$$f(\nu, \rho, \sigma) = f(\nu + 1, \rho, \sigma - 1). \tag{23}$$

It may be argued that if this surmise is correct our last modification has accomplished nothing from a practical point of view; we would have done just as well to use formula (20), replacing ν by $\nu+\sigma$. As an illustration of the method, however, it has some interest.

THE ROOTS OF A QUATERNION

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The theorems relative to the roots of a quaternion given by Niven in this MONTHLY, vol. 49, 1942, pp. 386–388, may be very simply obtained by writing a quaternion $q=d+ai+bj+ck$ in the form

$$q = k(\cos\theta + \epsilon \sin\theta), \qquad 0 \leqq \theta < 2\pi. \tag{1}$$

Here $k=\sqrt{d^2+a^2+b^2+c^2}$,

$$\cos\theta = \frac{d}{k}, \qquad \sin\theta = \pm\frac{\sqrt{a^2+b^2+c^2}}{k},$$

and in case $a^2+b^2+c^2 \neq 0$, ϵ is the unit vector

$$\epsilon = \pm\frac{ai+bj+ck}{\sqrt{a^2+b^2+c^2}}.$$

When q is real ($a=b=c=0$) ϵ may be chosen at pleasure.

Since $\epsilon^2 = -1$, we have by De Moivre's Theorem

$$q^n = k^n(\cos n\theta + \epsilon \sin n\theta). \tag{2}$$

If $q^n=Q$, Q will have the same unit vector as q. With Q given, we can always write

$$Q = K(\cos\phi + \epsilon \sin\phi), \qquad 0 \leqq \phi \leqq \pi. \tag{3}$$

Then, with the same ϵ for q, we have

$$k^n = K, \qquad \cos n\theta = \cos\phi, \qquad \sin n\theta = \sin\phi,$$

and the nth roots of Q are given by (1) provided

$$k = K^{1/n}, \qquad \text{the positive root}, \tag{4}$$

$$\theta = \frac{\phi}{n} + m\frac{2\pi}{n}, \qquad (m = 0, 1, \cdots, n-1). \tag{5}$$

These n values of θ comprise all values in the interval $0 \leqq \theta < 2\pi$ which satisfy the equations above.

When Q is *not real*, q has the same unit vector as Q and the n values of θ in (5) give precisely n quaternion nth roots.

When Q is *real* ϵ is arbitrary.

If $Q>0$, $\phi=0$ and

$$\theta = \frac{2m\pi}{n}, \qquad (m = 0, 1, \cdots, n-1).$$

When $n=2$, the values $\theta=0$, π give just two roots $\pm\sqrt{Q}$, both real. When $n>2$, some values of θ ($\neq 0$ or π) give non-real roots q and with these any unit vector ϵ may be associated.

If $Q<0$, $\phi=\pi$ and

$$\theta = \frac{(2m+1)\pi}{n}, \qquad (m = 0, 1, \cdots, n-1).$$

In every case some values of θ ($\neq\pi$) give non-real roots q and with these any unit vector ϵ may be associated.

We therefore have proved the

THEOREM. *A non-real quaternion has exactly n nth roots. If Q is real and positive it has just two square roots, $\pm\sqrt{Q}$; in all other cases a real quaternion has infinitely many roots.*

In order to *compute* the roots, formulas (4) and (5) suffice in all cases. For example if $Q=1+i+j+k$, we write

$$Q = 2(\cos 60° + \epsilon \sin 60°), \qquad \epsilon = \frac{i+j+k}{\sqrt{3}}.$$

The cube roots of Q are then

$$q = \sqrt[3]{2}\,(\cos\theta + \epsilon \sin\theta), \qquad \theta = 20°, 140°, 260°.$$

THE SPHERICAL PENDULUM AND COMPLEX INTEGRATION

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A century ago, Puiseux [1] published the following theorem: *The increment Φ of the azimuth of a spherical pendulum corresponding to its passage from the lowest level z_1 to the highest level z_2 is greater than $\pi/2$.* Later Halphen [2] found that Φ *is less than* π. Subsequently A. de Saint-Germain [3] and Gerard [4] gave simpler proofs for both inequalities, still involving somewhat laborious computations. Moreover, A. de Saint-Germain [3], following an idea of Hadamard [5], gave a simple and direct proof of Halphen's inequality based on the theory of residues.

In the present note complex integration will be used to give a simple demonstration of both inequalities. In the case of Halphen's inequality the argument reduces to that of Saint-Germain. The method allows a discussion of a new case.

It is well known that Φ is given by the formula

$$\Phi = \int_{z_1}^{z_2} \frac{lA\,dz}{(l^2 - z^2)\sqrt{(z - z_1)(z - z_2)(z - z_3)}} \tag{1}$$

where l is the length of the pendulum and z_1, z_2, z_3, satisfy the following conditions

$$-l < z_1 < z_2 < l < z_3, \qquad z_1 + z_2 < 0, \qquad z_3 = -\frac{l^2 + z_1 z_2}{z_1 + z_2},$$

so that the condition $z_3 > l$ is automatically satisfied. We put

$$A^2 = -\frac{(l^2 - z_1^2)(l^2 - z_2^2)}{z_1 + z_2} \tag{2}$$

and denote by A the positive square root of A^2. The polynomial $(z-z_1)(z-z_2)(z-z_3)$ takes for $z = \pm l$ the negative real value $-A^2$. We see that Φ depends only on the two real parameters z_1 and z_2 which are subject to the conditions $-l < z_1 < z_2 < l$ and $z_1 + z_2 < 0$.

Considering the integrand in (1) as a function of the complex variable $z = x + iy$, we obtain a single valued branch of this function in the domain exterior to the two narrow lanes (or cuts) C_1 and C_2 (see the Figure) by putting

$$z - z_k = r_k e^{i\theta_k}, \qquad 0 \leqq \theta_k < 2\pi, \qquad (k = 1, 2, 3). \tag{3}$$

With these conventions Φ is the positive value obtained by integration in the positive sense along the lower border of C_1. Our domain contains the two poles of our integrand $z = \pm l$; the corresponding residues are $i/2$, $i/2$. Applying Cauchy's theorem to our domain, Saint-Germain obtains immediately Halphen's inequality $\Phi < \pi$. It may be noticed that, by Cauchy's theorem, C_2 can be replaced by a *vertical line* L' cutting the real axis between l and z_3.

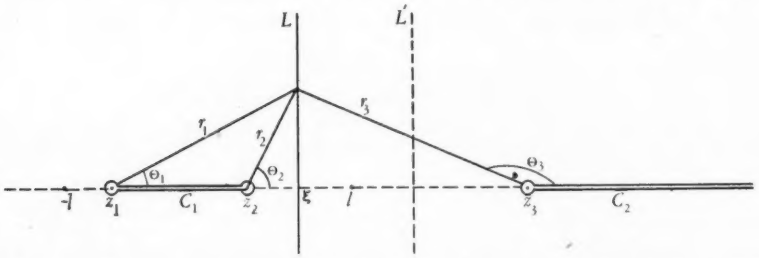
In order to obtain Puiseux's inequality $\Phi > \pi/2$ we shift L' to a new position L of abscissa $x=\xi$, $0 \leqq \xi < l_1$ and we apply Cauchy's theorem to the domain on the left of this vertical line L. We obtain at once

$$2\Phi = \pi + \int_{\xi-i\infty}^{\xi+i\infty} \tag{4}$$

where the integrand on the right hand side is the same as in (1). Since Φ and π are real the integral in (4) reduces to its real part. We have therefore, putting $\alpha = \alpha(y) = (\theta_1+\theta_2+\theta_3)/2$:

$$\int_{\xi-i\infty}^{\xi+i\infty} = \int_{-\infty}^{+\infty} \frac{ilAdy}{(l^2 - \xi^2 + y^2 - 2i\xi y)e^{i\alpha}\sqrt{r_1r_2r_3}}$$

$$= lA \int_{-\infty}^{+\infty} \left[\frac{(l^2 - \xi^2 + y^2)\sin\alpha - 2\xi y \cos\alpha}{(l^2 - \xi^2 + y^2)^2 + 4\xi^2y^2}\right] \frac{dy}{\sqrt{r_1r_2r_3}}.$$



In order to show that this integral is positive *let us take the non-negative abscissa ξ of L nearer to z_2 than to z_3*: $\xi - z_2 < z_3 - \xi$. Since $l < z_3$ and $-l < z_2 < l$ this condition can be fulfilled by non-negative values of ξ. We have then (see the Figure) for $y>0$: $\theta_2+\theta_3 > \pi$ and, *a fortiori*, $\theta_1+\theta_2+\theta_3 > \pi$. Moreover, we have obviously $\theta_1 < \pi/2$, $\theta_2 < \pi/2$, $\theta_3 \leqq \pi$ so that $\pi/2 < \alpha < \pi$ and the integrand is positive for $y>0$. Since $\alpha(-y) = 2\pi + \pi - \alpha(y)$ the integrand is an even function of y and therefore is positive also for $y<0$. In this way we obtain from (5) and (4), substituting the explicit value for the positive quantity A from (2),

$$\Phi = \pi/2 + l\sqrt{\frac{(l^2 - z_1^2)(l^2 - z^2)}{-(z_1+z_2)}} \int_0^\infty \frac{[(l^2 - \xi^2 + y^2)\sin\alpha - 2\xi y\cos\alpha]dy}{[(l^2 - \xi^2 + y^2)^2 + 4\xi^2y^2]\sqrt{r_1r_2r_3}} \tag{5}$$

which proves Puiseux's inequality $\Phi > \pi/2$.

By Cauchy's theorem we can shift now the abscissa ξ to any point of the interval $z_2 < \xi < l$. For instance, if z_1 and z_2 are both negative, we can take $\xi = 0$. Then by (6)

$$\Phi = \pi/2 + l\sqrt{\frac{(l^2 - z_1^2)(l^2 - z_2^2)}{-(z_1+z_2)}} \int_0^\infty \frac{\sin\alpha dy}{(l^2+y^2)\sqrt{r_1r_2r_3}}. \tag{7}$$

Consider now the case when z_1 and z_2 tend to $-l$, which implies that z_3 is converging to $l+0$. The integral in (7) remains finite and tends to a value J, so that we obtain $\lim \Phi = \pi/2$, a well known result which is usually obtained by somewhat delicate reasoning. The value J can be easily computed. Putting $z_1 = z_2 = -l$ we have

$$r_1 = r_2 = r_3,\ \theta_1 = \theta_2 = \pi - \theta_3,\ \alpha = \frac{\pi + \theta_1}{2},\ r_1 \cos\theta_1 = l,\ y = l\tan\theta_1,\ l^2 + y^2 = r_1^2.$$

Therefore we have

$$J = \int_0^\infty \frac{\cos\dfrac{\theta_1}{2}\,dy}{r_1^3\sqrt{r_1}} = \frac{1}{l^2\sqrt{l}}\int_0^{\pi/2} \cos\frac{\theta_1}{2}\cos\theta_1\sqrt{\cos\theta_1}\,d\theta_1$$

or, putting $\zeta = \sqrt{2}\sin\theta_1/2$,

$$J = \frac{\sqrt{2}}{l^2\sqrt{l}}\int_0^1 (1-\zeta^2)\sqrt{1-\zeta^2}\,d\zeta = \frac{\sqrt{2}}{l^2\sqrt{l}}\,\frac{3\pi}{16}.$$

Since $-(z_1+z_2)$ tends to $2l$, we have finally, from (6)[7]

$$\lim\left(\Phi - \frac{\pi}{2}\right)(l^2 - z_1^2)^{-1/2}(l^2 - z_2^2)^{-1/2} = 3\pi/16.$$

Using again (6), we see that Φ converges to $\pi/2$ when z_1 tends to $-l$ and z_2 remains arbitrarily fixed. In this case the motion approximates the oscillations of a simple pendulum of finite amplitude. It seems that this case has not been mentioned in the bibliography of the question. Let us also note that the method of complex integration allows to represent the excess of Φ over $\pi/2$ explicitly in the form of an integral.

References

1. Puiseux, Journal de Liouville, 7, 1842, p. 517.
2. Halphen, Traité des Fonctions Elliptiques, II, p. 128.
3. A. de Saint-Germain, Bulletin des Sciences Mathématiques, (2), 20, 1896, p. 114 and (2), 22, 1898, p. 95.
4. L. Gérard, Nouvelles Annales de Mathématiques, (6), Vol. II, 1927, p. 147.
5. J. Hadamard, Bulletin des Sciences Mathématiques, (2), 19, 1895, p. 228. Reference to this paper, concerning a similar question in the theory of tops, can be found in W. D. MacMillan, Dynamics of Rigid Bodies, 1936, p. 233.
6. P. Appell, Traité de Mécanique Rationelle, I, p. 517.
7. Cf. J. L. Synge and B. A. Griffith, Principles of Mechanics, 1942, p. 374.

THE HAGGE CIRCLE OF A TRIANGLE

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1. Introduction. If from the vertices A_i, $i=1, 2, 3$, of a triangle, lines are drawn through a fixed point P, meeting the circumcircle again at B_i, and if C_i is the reflection of B_i in A_jA_k, then the points C_i determine the *Hagge Circle* of P with respect to triangle $A_1A_2A_3$ [1, p. 300]. We denote by $H(P, T)$ the Hagge circle of P with respect to triangle T.

In the following discussion we shall use conjugate coördinates,* taking as the vertices of our triangle T, the turns t_i, $i=1, 2, 3$; *i.e.*, complex numbers with unit modulus. Then the origin of coördinates, O, is the circumcenter of T.

Consider a fixed point p not on the circumcircle of T, so that p is not a turn. Then the points C_i which determine $H(p, T)$ are readily found to be

$$c_i = t_j + t_k - \frac{t_jt_k(t_i\bar{p} - 1)}{t_i - p}. \tag{1}$$

Introducing the symmetric functions

$$S_1 = t_1 + t_2 + t_3,$$
$$S_2 = t_1t_2 + t_2t_3 + t_3t_1,$$
$$S_3 = t_1t_2t_3,$$

we may write (1) in the form

$$c_i = \frac{S_2 - S_3\bar{p} - pS_1 + pt_i}{-p + t_i}. \tag{2}$$

If we replace t_i by the variable turn t, we obtain

$$x = \frac{S_2 - S_3\bar{p} - pS_1 + pt}{-p + t} \tag{3}$$

which is the equation of a circle through the points c_i, or precisely $H(p, T)$. If, in particular, we let $t=(S_2-S_3\bar{p})/(S_1-p)$ we have $x=S_1$. But S_1 is the orthocenter of T [4], so that *the Hagge circle of any point with respect to a given triangle passes through the orthocenter of the triangle* [1, p. 300].

Eliminating t from (3) and from the conjugate equation of (3), we have the equation of $H(p, T)$ in the form

$$(x - c)(\bar{x} - \bar{c}) = q\bar{q} \tag{4}$$

where

$$c = \frac{p - S_1p\bar{p} + S_2\bar{p} - S_3\bar{p}^2}{1 - p\bar{p}} \tag{5}$$

* For explanation of this method, see [2], [3] and [4].

and

$$= \frac{S_3\bar{p}^2 - S_2\bar{p} + S_1 - p}{1 - p\bar{p}}; \tag{6}$$

that is, $H(p, T)$ has center at c, and radius $\sqrt{q\bar{q}}$. But from (6), q is the isogonal conjugate [4] of p in T. Hence, *the radius of the Hagge circle of p is the distance from the circumcenter to the isogonal conjugate of p.*

It follows immediately from (5) and (6), that

$$(c + q)/2 = S_1/2. \tag{7}$$

Now $S_1/2$, being midway between the circumcenter and the orthocenter of T, is the nine-point center of T, so that *for any point p, the nine-point center is midway between the Hagge center and the isogonal conjugate of p.*

If we allow p to be on the circumcircle of T, but not at t_i, then $H(p, T)$ is simply the line of images [5] of p with respect to T.

2. Hagge circles in an orthocentric system. Let p be any point not a member of the orthocentric system t_1, t_2, t_3, S_1, and consider $H(p, T_i)$ where T_i is the triangle $t_jt_kS_1$. Let $m = S_1 - p$. Then the isogonal conjugate of p in T_i is found to be

$$q_i = S_1 - \frac{t_i\bar{m}(S_3\bar{m} - S_2 + t_im - t_i^2)}{(t_im\bar{m} - m - t_i^2\bar{m})}.$$

Since the nine-point circle of T is also the nine-point circle of T_i [1, p. 197], it follows from (7) that the center of $H(p, T_i)$ is given by

$$d_i = \frac{t_i\bar{m}(S_3\bar{m} - S_2 + t_im - t_i^2)}{(t_im\bar{m} - m - t_i^2\bar{m})}.$$

Now the circle we are seeking has center at d_i and passes through t_i, the orthocenter of T_i. We find its second intersection with the circumcircle of T to be $E = d_i/(t_i\bar{d}_i) = S_3\bar{m}/m$. And so

$$E = \frac{S_2 - S_3\bar{p}}{S_1 - p}.* \tag{8}$$

Since this result is symmetric in t_1, t_2, t_3, we have

THEOREM 1. *The Hagge circles of any point distinct from the points of an orthocentric system, with respect to three triangles of the system, are concurrent at a point on the circumcircle of the fourth triangle.*

It follows from (8) that $Ep - S_3\bar{p} = S_1E - S_2$. The set of points p which satisfy this equation is precisely the line of images [5] of the point E with respect to T.

* Other properties of the point E were first stated by J. R. Musselman, [5], [6].

Conversely, for any point p on the line of images of E, but not at the orthocenter S_1, (8) must hold. Hence we have

THEOREM 2. *Let T, T_1, T_2, T_3 be the triangles of an orthocentric system, and let E be any point on the circumcircle of T. Then the circles $H(P, T_i)$, $i=1, 2, 3$, where P is not the orthocenter of T, pass through E if and only if P is on the line of images of E with respect to T.*

3. Hagge circles in the Brocard configuration. Let the circles tangent at t_i to t_it_j and t_it_k respectively meet again at b_i. Then b_i, $i=1, 2, 3$, are the vertices of the *second Brocard triangle* of T [1, p. 279]. We easily obtain

$$b_i = \frac{S_2 - S_1 t_i}{S_1 - 3t_i}. \tag{9}$$

The circle determined by the b_i is the *Brocard Circle* of T, and its equation may be obtained from (9) in exactly the same manner in which (4) was obtained from (1), (2) and (3):

$$x\bar{x} - \left(\frac{3S_2 - S_1^2}{9S_3 - S_1S_2}\right)x - \left(\frac{3S_1S_3 - S_2^2}{9S_3 - S_1S_2}\right)\bar{x} = 0. \tag{10}$$

The transformation

$$x = S_1 - 2x' \tag{11}$$

sends triangle T into its medial triangle T', and sends (4) into

$$x\bar{x} - \frac{\bar{q}}{2}x - \frac{q}{2}\bar{x} = 0, \tag{12}$$

a circle through the circumcenter of T, *i.e.*, through the orthocenter of T'; q here is defined as in (6). If, in particular, we take p as the median point of T, and hence, as the median point of T', we have $p=S_1/3$, and (12) reduces immediately to (10). Hence, *in any triangle, the Hagge circle of the median point with respect to the medial triangle is the Brocard circle of the given triangle* [1, p. 300].

We consider the points c_i' which determine the Brocard circle of T as $H(S_1/3, T')$. Putting $p=S_1/3$ in (2), we have

$$c_i = \frac{2S_2 - S_1^2 + S_1t_i}{3t_i - S_1}. \tag{13}$$

Applying (11), and noticing that this leaves $S_1/3$ invariant, we find that the desired points are

$$c_i' = \frac{S_2 - S_1t_i}{S_1 - 3t_i}. \tag{14}$$

Comparing (9) and (14), we have

THEOREM 3. *The points which determine the Brocard circle of any triangle as a Hagge circle of its medial triangle are the vertices of the second Brocard triangle of the given triangle.*

References

1. R. A. Johnson, Modern Geometry, Houghton Mifflin Co., 1929.
2. H. A. DoBell, On the geometry of the triangle, this MONTHLY, vol. 39, 1932, pp. 71–85.
3. J. H. Weaver, Curves determined by a one-parameter family of triangles, this MONTHLY, vol. 40, 1933, pp. 85–91.
4. J. H. Weaver, On isogonal points, this MONTHLY, vol. 42, 1935, pp. 496–499.
5. J. R. Musselman, On the line of images, this MONTHLY, vol. 45, 1938, pp. 421–430.
6. J. R. Musselman, Some loci connected with a triangle, this MONTHLY, vol. 47, 1940, pp. 354–361.

CONVEX SETS*

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The systematic study of convex sets was begun by Brunn in 1887 and carried on by Minkowski and others whose works are listed in the bibliography of Bonnesen and Fenchel.† The elementary plane of support properties of convex sets are very well known. However, about the only readily accessible discussion of these properties is that of L. L. Dines which appeared in this MONTHLY in April, 1938. The proof there given‡ for the fundamental theorem, that through any given boundary point of a convex set there passes a plane supporting the set, depends on exhibiting a sequence of bounding planes whose limit, in the sense there defined, turns out to be the desired plane of support. In the present note we present two distinct, and, we believe, new proofs of this theorem neither of which involves any such limiting process. One of these proofs is due to Professor McShane,§ and we wish to thank him for permission to include it here.

The last section contains some simple related theorems on the existence and continuous turning of planes tangent to convex surfaces. Though these theorems seem interesting and non-trivial, so far as we know no one has hitherto bothered to record them.‖

* Presented at the Virginia-District of Columbia-Maryland section meeting of the Mathematical Association of America, May, 1940.

† Bonnesen and Fenchel, Theorie der Konvexen Körper, Berlin, 1934.

‡ Due, as the author remarks, to Carathéodory.

§ This proof will also appear shortly in a paper by Professor McShane in the Revista de Ciencias.

‖ We have been informed that the two dimensional case is discussed by J. Hjelmslev in Contribution à la géometrié infinitésimale de la courbe réelle, Overs. Danske Vidensk. Selsk. Forh., 1911, p. 433–494.

1. Notation. The space with which we shall deal is an n dimensional euclidean space E_n. In cartesian coördinates the points of E_n are the vectors $\mathbf{x}=(x^1, \cdots, x^n)$ with real components. By the dimension of any point set A in E_n we mean the dimension of the subspace of E_n of lowest dimension containing A. Repetition of the superscript i in any term will require summation over the values $1, \cdots, n$. Thus for the distance of the point $\mathbf{x}$ from the origin we write

$$|\mathbf{x}| = [x^i x^i]^{1/2} = [(x^1)^2 + \cdots + (x^n)^2]^{1/2}.$$

The linear equation $x^i u^i - u^0 = 0$, u^0 being a constant and $\mathbf{u}=(u^1, \cdots, u^n)$ a fixed non-null vector, defines a *hyperplane* or as we shall say simply a *plane* in E_n. The vector $\mathbf{u}$ defines the normal direction of the plane. If $\mathbf{x}_0, \cdots, \mathbf{x}_p$ are $p+1$ points lying in no $p-1$ dimensional subspace of E_n, the set of points of the form

$$\mathbf{x} = \lambda_0 \mathbf{x}_0 + \cdots + \lambda_p \mathbf{x}_p, \quad \lambda_0 + \cdots + \lambda_p = 1,$$
$$\lambda_j \geqq 0 \qquad (j = 0, \cdots, p)$$

is called the p-simplex $S(x_0, \cdots, x_p)$. Clearly the dimension of a p-simplex is p.

If there exists a neighborhood of a point $\mathbf{x}$ which is contained in a set A, the point $\mathbf{x}$ is said to be *interior* to A. If every neighborhood of a point $\mathbf{x}$ contains both points in A and points not in A, then $\mathbf{x}$ is called a *boundary point* of A. It can easily be shown that

(1.1) *a simplex S in E_n has an interior point if and only if S is an n-simplex.*

We now proceed to the definition and immediate properties of convex sets.

2. Convexity. A set K in E_n is said to be *convex* provided for every pair of points $\mathbf{x}=(x^1, \cdots, x^n)$, $\mathbf{y}=(y^1, \cdots, y^n)$ in K all points of the line segment

$$(1-\lambda)\mathbf{x} + \lambda\mathbf{y} \qquad (0 \leqq \lambda \leqq 1)$$

joining $\mathbf{x}$ and $\mathbf{y}$ are in K. A closed* convex set is called a *convex body*.

The following properties of convex sets are easily established.

(2.1) *If a convex set K contains the vertices of a simplex S, then K contains all the points of S.*

(2.2) *The closure† $\overline{K}$ of a convex set K is a convex body.*

(2.3) *The intersection‡ of any collection of convex sets is a convex set.*

If a convex set K is n dimensional, it contains the vertices of an n-simplex S, and hence by (2.1) K contains all of S. By (1.1) S has an interior point. Therefore

* A set which contains all its limit points is called closed.

† The closure $\bar{A}$ of a set A is the set consisting of all the points of A and all the limit points of A.

‡ The intersection of a collection of sets is the set of all points x such that x is in every set of the collection.

(2.4) *a convex set K in E_n has an interior point if (and only if) K is n dimensional.*

(2.5) *If the point $\mathbf{x}$ is interior to the convex set K and $\mathbf{y}$ is any point of K, then all points of the form*

$$(1-\lambda)\mathbf{x}+\lambda\mathbf{y} \qquad (0 \leqq \lambda < 1)$$

are interior to K.

(2.6) *If $\mathbf{y}$ is a point external to the convex body K, there is a unique point $\mathbf{x}_0$ of K closest to y.*

A plane $\pi: x^i u^i - u^0 = 0$ *bounds* a set A provided all points of A lie together in just one of the open half-spaces $x^i u^i - u^0 < 0$ and $x^i u^i - u^0 > 0$ defined by π. A plane π is said to be a *plane of support* of the set A provided the intersection $\pi \cdot A$ is non-empty and π bounds the set $A - \pi \cdot A$ of points of A which are not in π. A plane π *separates* two sets A and B provided A and B lie one in each of the two open half-spaces of π.

THEOREM 2.1. *If $\mathbf{y}$ is a point external to a convex body K and $\mathbf{x}_0$ is the point of K closest to $\mathbf{y}$, then the plane π passing through $\mathbf{x}_0$ with normal defined by the vector $\mathbf{y}-\mathbf{x}_0$ is a plane of support of K separating y from the set $K-\pi \cdot K$.*

This is seen intuitively as follows. Suppose there is a point $\mathbf{x}_1$ of K on the same side of π as $\mathbf{y}$. Then there are points on the line segment $\mathbf{x}_0\mathbf{x}_1$ closer to $\mathbf{y}$

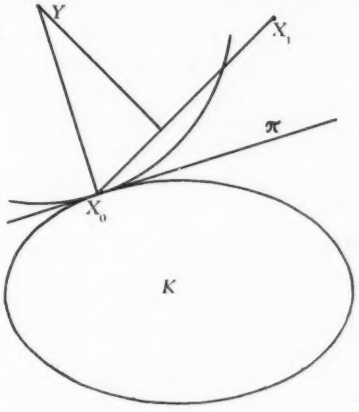


FIG. 1

than $\mathbf{x}_0$, and these points are in K by convexity. This contradicts the choice of $\mathbf{x}_0$ as the point of K closest to $\mathbf{y}$.

For notational convenience in proving the theorem we take $\mathbf{y}$ to be the origin. Then the plane π has the equation

$$x^i x_0^i - x_0^i x_0^i = 0.$$

Let $\mathbf{z}=(z^1, \cdots, z^n)$ by any point of K. Define

$$\phi(\lambda) = \left| \mathbf{x}_0 + \lambda(\mathbf{z} - \mathbf{x}_0) \right|^2.$$

Since the line segment $\mathbf{z}\mathbf{x}_0$ is contained in K and $\mathbf{x}_0$ is the point of K closest to $\mathbf{y}$, on the interval $0 \leqq \lambda \leqq 1$ the function $\phi(\lambda)$ has its least value when $\lambda = 0$. Hence

$$\phi'(0) \geqq 0.$$

Therefore if we set $\lambda = 0$ in the equation

$$\phi'(\lambda) = 2x_0^i(\mathbf{z} - \mathbf{x}_0)^i + 2\lambda \left| \mathbf{z} - \mathbf{x}_0 \right|^2,$$

it follows that

$$z^i x_0^i - x_0^i x_0^i \geqq 0.$$

And now since

$$y^i x_0^i - x_0^i x_0^i = - x_0^i x_0^i < 0,$$

the proof is complete.

COROLLARY 2.1. *Under the hypotheses of Theorem* 2.1 *there are planes separating* $\mathbf{y}$ *from* K *and there is a plane passing through* $\mathbf{y}$ *bounding* K.

THEOREM 2.2. *Let* A *be a closed set having an interior point* $\mathbf{x}$. *If through each boundary point of* A *there passes a plane of support of* A, *then* A *is convex.*

In case A is the entire space the conclusion is obvious. Otherwise, on the line segment joinging $\mathbf{x}$ and any point $\mathbf{y}$ not in A there is a boundary point $\mathbf{z}$ of A. By hypothesis there is a plane of support π of A passing through $\mathbf{z}$. The plane π cannot contain $\mathbf{y}$ else it would also contain $\mathbf{x}$, since $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are collinear and distinct. Hence the closed half-space of π containing $\mathbf{x}$ contains A but not $\mathbf{y}$. Therefore the intersection Π of all such closed half-spaces contains A but no point external to A. That is, $\Pi = A$. And by (2.3) Π is convex.

3. The fundamental theorem. In order to establish a modified converse of Theorem 2.2 we first consider a particular kind of convex set, the convex cone. A convex body C is called a *convex cone* with the point $\mathbf{x}_0$ as *vertex* provided C contains a point $\mathbf{x}$ different from $\mathbf{x}_0$ and for each such point $\mathbf{x}$ the set C contains the entire ray

$$(1 - \lambda)\mathbf{x}_0 + \lambda\mathbf{x} \qquad (\lambda \geqq 0).$$

THEOREM 3.1. *Every convex cone* C *which is not the whole space has at least one plane of support, and every plane of support of* C *passes through the vertex* $\mathbf{x}_0$ *of* C.

That C has a plane of support $\pi: x^i u^i - u^0 = 0$ follows from Theorem 2.2. To show that π must necessarily pass through $\mathbf{x}_0$ consider any point $\mathbf{x}_1$ in the set $\pi \cdot C$. We may as well suppose that C lies in the positive closed half-space of π. Then

$$[(1 - \lambda)\mathbf{x}_0 + \lambda\mathbf{x}_1]^i u^i - u^0 \geqq 0$$

for all $\lambda \geqq 0$. And now since $x_1^i u^i - u^0 = 0$, we have

$$(1 - \lambda)(x_0^i u^i - u^0) \geqq 0 \qquad (\lambda \geqq 0).$$

Since the first factor in this expression can change sign while the second cannot, we must have $x_0^i u^i - u^0 = 0$.

If $\mathbf{x}_0$ is a boundary point of a convex body K, let T be the set of all points lying on rays from $\mathbf{x}_0$ through the points of K.

LEMMA 3.1. *The closure $\overline{T}$ of the set T is a convex cone.*

By (2.2), to show that the set $\overline{T}$ is a convex body we need only show that the set T is convex. Taking $\mathbf{x}_0$ to be the origin, let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any two points of T. We must show that any point*

$$\mathbf{x} = \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2, \quad \lambda_1 > 0, \quad \lambda_2 > 0, \quad \lambda_1 + \lambda_2 = 1,$$

of the line segment $\mathbf{x}_1\mathbf{x}_2$ is also in T. By definition of T there exist positive numbers m_1 and m_2 such that the points $\mathbf{y}_1 = m_1\mathbf{x}_1$ and $\mathbf{y}_2 = m_2\mathbf{x}_2$ are in K. Writing

$$m = \left[\frac{\lambda_1}{m_1} + \frac{\lambda_2}{m_2}\right]^{-1},$$

we see that the point

$$\begin{aligned} \mathbf{y} &= m\mathbf{x} \\ &= m(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2) \\ &= m\left(\frac{\lambda_1}{m_1}\mathbf{y}_1 + \frac{\lambda_2}{m_2}\mathbf{y}_2\right). \end{aligned}$$

is on the line segment $\mathbf{y}_1\mathbf{y}_2$ and hence is in K. Therefore the point $\mathbf{x} = m^{-1}\mathbf{y}$ is in T by definition of T.

It remains only to show that if $\mathbf{z}$ is any point of the set $\overline{T} - T$, then any point

$$\lambda\mathbf{z}, \qquad (\lambda > 0)$$

of the ray from $\mathbf{x}_0$ through $\mathbf{z}$ is in T. Choose any positive number ϵ. There is a point $\mathbf{x}$ of T such that

$$|\mathbf{z} - \mathbf{x}| < \frac{\epsilon}{\lambda}.$$

Hence

$$|\lambda\mathbf{z} - \lambda\mathbf{x}| < \epsilon,$$

and now since the point $\lambda\mathbf{x}$ is in T, $\lambda\mathbf{z}$ is in $\overline{T}$.

* This notation is seen to be equivalent to that previously used if we write $\lambda = \lambda_1$, $1 - \lambda = \lambda_2$.

THEOREM 3.2. *If $\mathbf{x}_0$ is a boundary point of a convex body K in E_n then there is a plane of support of K passing through $\mathbf{x}_0$.*

This will be an immediate consequence of Lemma 3.1 and Theorem 3.1 provided it can be shown that the set $\overline{T}$ of Lemma 3.1 is not the entire space. This is surely true if K is not n-dimensional. If K is n-dimensional, K has an interior point $\mathbf{x}$ by (2.5). Take $\mathbf{x}_0$ as the origin. Suppose $\overline{T}$ *is* the whole space. Then in particular the point $-\mathbf{x}$ is in $\overline{T}$. Hence there is a point $-\mathbf{y}$ of T so close to $-\mathbf{x}$ that $\mathbf{y}$, being equally close to $\mathbf{x}$, is interior to K. By definition of T, on the ray from $\mathbf{x}_0$ through $-\mathbf{y}$ there is a point $\mathbf{z}$ of K different from $\mathbf{x}_0$. But now by (2.5) $\mathbf{x}_0$, being on the segment $\mathbf{yz}$, is interior to K, which is a contradiction.

Professor McShane's proof of Theorem 3.2, which we mentioned in the introduction, is as follows. Let the given boundary point $\mathbf{x}_0$ of the convex body K

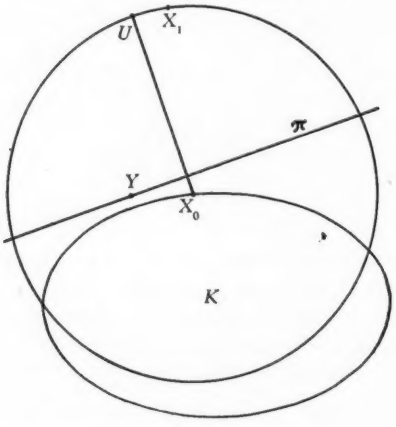


FIG. 2

be the origin, and let S be the sphere $|\mathbf{x}| = 1$ about $\mathbf{x}_0$. The distance from K to any point $\mathbf{x}$ of S, being a continuous function of $\mathbf{x}$ on S, attains its maximum at some point $\mathbf{x}_1$ exterior to K. We shall now show that $\mathbf{x}_0$ is the point of K closest to $\mathbf{x}_1$.

Choose any positive number $\epsilon < 1$ and any point $\mathbf{y}$ exterior to K for which $|\mathbf{y}| < \epsilon$. By Corollary 2.1 there is a plane $\pi: x^i u^i - u^0 = 0$ passing through $\mathbf{y}$ and bounding K. If we suppose, as we may, that the magnitude of the vector $\mathbf{u} = (u^1, \cdots, u^n)$ is 1, the distance from the plane π to the point $\mathbf{x}_0$ is $|u^0|$. And since $\mathbf{y}$ is in this plane and $|\mathbf{y}| < \epsilon$, it follows that

$$|u^0| < \epsilon.$$

Hence for the distance $1 - u^0$ from the plane to the point $\mathbf{u}$ of S the inequality

$$1 - u^0 > 1 - \epsilon$$

holds. And now since the point $\mathbf{u}$ and the set K are on opposite sides of the plane π, the distance from $\mathbf{u}$ to K is greater than $1-\epsilon$. We have thus shown that for every positive number $\epsilon<1$ there exists a point $\mathbf{u}$ of S whose distance from K is greater than $1-\epsilon$. Hence the distance of the point $\mathbf{x}_1$ from K cannot be less than 1, which is the distance of $\mathbf{x}_1$ from $\mathbf{x}_0$. It now follows from Theorem 2.1 that there is a plane of support of K passing through the point $\mathbf{x}_0$.

4. Planes tangent to convex surfaces. The proof usually given for Theorem 3.2* is essentially contained in the following lemma which we shall find useful in another connection and which we shall state without proof. We say that the sequence of planes $x^i u_j^i - u_j^0 = 0$, $\left|(u_j^1, \cdots, u_j^n)\right| = 1$, $j = 1, 2, \cdots$, converges to the plane $x^i u^i - u^0 = 0$ provided the sequence of points $(u_j^0, u_j^1, \cdots, u_j^n)$ in $n+1$ space converges to the point $(u^0, u^1, \cdots, u^n)$.

LEMMA 4.1. *Let $\{\pi_j\}$ denote a sequence of planes, each either bounding or supporting the closed set A, such that for some boundary point $\mathbf{x}_0$ of A the distance from the plane π_j to $\mathbf{x}_0$ tends to zero as j tends to infinity. Then there exists a subsequence of $\{\pi_j\}$ converging to a plane π, and for each such convergent subsequence the limiting plane π passes through the point $\mathbf{x}_0$ and supports the set A.*

We call a set S in E_n a surface provided for every point $\mathbf{x}$ in S there is an arbitrarily small neighborhood N of $\mathbf{x}$ such that the intersection $N \cdot S$ is homeomorphic† with the interior of an $(n-1)$ dimensional sphere whose center corresponds to the point $\mathbf{x}$. It is not difficult to show that the set of boundary points of a convex body with interior points is a surface in this sense. Let $\rho(\mathbf{x}, \pi)$ denote the distance from the point $\mathbf{x}$ to the plane π. A plane π is said to be tangent to a surface S at a point $\mathbf{x}_0$ of S provided for any sequence $\mathbf{x}_1, \mathbf{x}_2 \cdots$ of points of S which converges to $\mathbf{x}_0$ the sequence of numbers

$$\frac{\rho(\mathbf{x}_1, \pi)}{|\mathbf{x}_1 - \mathbf{x}_0|}, \quad \frac{\rho(\mathbf{x}_2, \pi)}{|\mathbf{x}_2 - \mathbf{x}_0|}, \cdots$$

converges to zero. Clearly when a tangent plane exists it is unique.

THEOREM 4.1. *The boundary surface S of a convex body K with interior points has a tangent plane at a point $\mathbf{x}_0$ of S if and only if the plane of support of K passing through $\mathbf{x}_0$‡ is unique. In this case the plane of support is the tangent plane.*

Assuming first that S has the tangent plane τ at $\mathbf{x}_0$, suppose there is a plane π distinct from τ, which supports K at $\mathbf{x}_0$. For convenience take $\mathbf{x}_0$ as the origin and τ as the plane $x^1 = 0$. Choose any point $\mathbf{y} = (y^1, \cdots, y^n)$ which is interior to K and not in τ. Say for definiteness that $y^1 > 0$. Choose any point $\mathbf{z} = (z^1, \cdots, z^n)$

* See the introduction and the references there given.

† Two sets are said to be homeomorphic with each other provided there is a one to one correspondence between their points which, considered as a transformation from either set to the other, is continuous.

‡ Which exists by Theorem 3.2.

of π for which $z^1>0$. Such points clearly exist, since the planes π and τ are distinct. For $j=1, 2, \cdots$, the points $2^{-j}\mathbf{y}$ are interior to K (by (2.5)) while the points $2^{-j}\mathbf{z}$ are not. Hence for each value of j there is a point in S of the form

$$\mathbf{x}_j = (1-\lambda_j)2^{-j}\mathbf{y} + \lambda_j 2^{-j}\mathbf{z}, \qquad 0 < \lambda_j \leqq 1.$$

The distance of the point $\mathbf{x}_j$ from the plane τ is x_j^1. Since for all j

$$\frac{x_j^1}{|\mathbf{x}_j|} = \frac{(1-\lambda_j)y^1 + \lambda_j z^1}{|(1-\lambda_j)\mathbf{y} + \lambda_j \mathbf{z}|} \geqq \frac{\min\,[y^1, z^1]}{|\mathbf{y}| + |\mathbf{z}|} > 0,$$

we have

$$\lim_{j\to\infty} \frac{x_j^1}{|\mathbf{x}_j|} > 0,$$

which contradicts the tangency of the plane τ to the surface S. This establishes half of the theorem.

Assume now that K has the unique plane of support π at $\mathbf{x}_0$. Let $\mathbf{x}_0$ be the origin, let $x^1=0$ be the equation of the plane π, and let K lie in the half space $x^1 \geqq 0$. If π is not tangent to S at $\mathbf{x}$, we can choose a positive number ϵ and a sequence of points $\{\mathbf{x}_j\}$ of S converging to $\mathbf{x}_0$ such that for all j

$$\frac{x_j^1}{|\mathbf{x}_j|} \geqq \epsilon.$$

By the Bolzano-Weierstrass theorem we may suppose the sequence $\{\mathbf{x}_j\}$ so chosen that the sequence of unit vectors

$$\left\{\frac{\mathbf{x}_j}{|\mathbf{x}_j|}\right\}$$

converges. By Theorem 3.2, through each point $\mathbf{x}_j$ there passes a plane $\pi_j: x^i u_j^i - u_j^0 = 0$ supporting K. For each j we may suppose that the magnitude of the vector $(u_j^1, \cdots, u_j^n)$ is 1 and that for all points $\mathbf{x}$ of K

$$x^i u_j^i - u_j^0 \geqq 0.$$

Further, by Lemma 4.1 we may suppose that the planes π_j converge to a plane $\pi_0: x^i u_0^i - u_0^0 = 0$ supporting K at the point $\mathbf{x}_0$. The number u_0^0 is zero since x_0 is the origin. Now if $\mathbf{u}$ is the vector $(1, 0, \cdots, 0)$,

$$\frac{x_j^i}{|\mathbf{x}_j|} u^i = \frac{x_j^1}{|\mathbf{x}_j|} \geqq \epsilon.$$

Therefore

$$\lim_{j\to\infty} \frac{x_j^i}{|\mathbf{x}_j|} u^i \geqq \epsilon > 0. \tag{4.1}$$

Since the point $\mathbf{x}_j$ is in the plane π_j,

$$x_j^i u_j^i - u_j^0 = 0,$$

and since the origin is in K,

$$-u_j^0 \geqq 0.$$

Hence we have

$$\frac{x_j^i}{|\mathbf{x}_j|} u_j^i \leqq 0.$$

Letting j tend to infinity in this expression, we get

$$\lim_{j\to\infty} \frac{x_j^i}{|\mathbf{x}_j|} u_0^i \leqq 0. \tag{4.2}$$

Comparing (4.1) and (4.2) we see that the unit vectors $\mathbf{u}$ and $\mathbf{u}_0$ must be different. Since K is n dimensional and all points of K satisfy both the relations

$$x^i u^i \geqq 0, \qquad x^i u_0^i \geqq 0,$$

it follows that u is also different from $-u_0$. Hence the planes $\pi: x^i u^i = 0$ and $\pi_0: x^i u_0^i = 0$ are distinct, contradicting the hypothesis that π is the only plane of support of K passing through the point $\mathbf{x}_0$. The proof of Theorem 4.1 is now complete.

We say that a plane $x^i u^i(y) - u^0(y) = 0$, $|u^1(y), \cdots, u^n(y)| = 1$, *varies continuously* in $\mathbf{y}$ on some set in n-space provided the vector

$$\mathbf{u}(\mathbf{y}) = (u^0(\mathbf{y}), u^1(\mathbf{y}), \cdots, u^n(\mathbf{y}))$$

is a continuous function of $\mathbf{y}$ on that set.

THEOREM 4.2. *If the boundary surface S of a convex body K with interior points has a tangent plane $\pi(x)$ at each point $\mathbf{x}$ of S which is in a given neighborhood N in S, then this plane varies continuously in $\mathbf{x}$ on N.*

By Theorem 4.1 at each point $\mathbf{x}$ of N the set K has the unique plane of support $\pi(\mathbf{x})$. If $\pi(\mathbf{x})$ is discontinuous at some point $\mathbf{x}_0$ of N, there is a sequence of points $\{\mathbf{x}_j\}$ of N converging to $\mathbf{x}_0$ such that the sequence of planes $\pi(\mathbf{x}_j)$ does not converge to the plane $\pi(\mathbf{x}_0)$. From these we may select a subsequence convergent to a plane π^* distinct from $\pi(\mathbf{x}_0)$. By Lemma 4.1 the plane π^* passes through the point $\mathbf{x}_0$ and supports the set K. This contradicts the uniqueness of the plane of support $\pi(\mathbf{x}_0)$.

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The Department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A THEOREM ON IMPROPER INTEGRALS

ARTHUR SARD, Queens College

Suppose that the function $f(x)$ is bounded and integrable in the sense of Riemann in each interval $[a, t]$, where a is fixed and $a \leqq t < \infty$. Consider the integral

$$I = \int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

and the series

$$S = \sum_{i=1}^{\infty} f(x_i')(x_{i+1} - x_i),$$

where

$$\sigma = \{x_1, x_1', x_2, x_2', x_3, x_3', \cdots\}$$

is a non-decreasing unbounded sequence with $x_1 = a$ such that Δ, the least upper bound of the differences $x_{i+1} - x_i (i = 1, 2, \cdots)$, is at most 1. Consider also

$$L = \lim_{\Delta\to 0} S.$$

In general any one of the limits I, S or L need not exist.

THEOREM. *The limits I, S and L exist and are finite, and $L = I$ providing that $\sum_{j=1}^{\infty} M_j < \infty$, where M_j is the least upper bound of $|f(x)|$ for x in the closed interval $[a+j-1, a+j]$, $(j = 1, 2, \cdots)$.*

The proof of the theorem is given below.

COROLLARY. *If $f(x)$ is non-negative and non-increasing in $a \leqq x < \infty$, the mere convergence of I implies that S and L exist and that $L = I$.*

Proof: In this case the convergence of I implies the hypothesis of the theorem, by the integral test for the convergence of a series.*

* D. R. Dickinson has stated the following theorem, which can be considered a corollary of our theorem. *If there exists a non-increasing function $g(x)$ such that $|f(x)| \leqq g(x)$ for all $x \geqq a$ and such that $\int_a^\infty g(x)dx < \infty$, then S, L, and I exist and $L = I$.* (Quarterly Journal of Mathematics, Oxford Series, vol. 12, 1941, pp. 176–183). The present author, at the time he submitted the manuscript to the editor, was unaware of Dickinson's work.

In general the convergence of I does not even imply that S exists. To see this consider a function $f(x)$ of the following type: $f(x)$ is non-negative and continuous; the graph of $f(x)$ between $x=j-1$ and $x=j$ consists of 2^{j-1} congruent peaks each of area $1/2^{2j-1}$ and of maximum height $1/j$ $(j=1, 2, \cdots)$. Then $I=1$. Nevertheless S can diverge to ∞ with Δ arbitrarily small. Thus, let the unprimed elements of the sequence σ include all but a finite number of the positive integers and let all but a finite number of the primed elements of σ be such as to afford relative maxima to $f(x)$. Then comparison with the harmonic series shows that $S=\infty$.

Proof of the theorem: Consider the series

$$I' = \sum_{j=1}^{\infty} \int_{a+j-1}^{a+j} f(x)dx$$

and

$$S' = \sum_{j=1}^{\infty} u_j,$$

where $u_j=\sum f(x_i')(x_{i+1}-x_i)$ for all i such that $a+j-1 \leqq x_i < a+j$.

Comparison of S' with $\sum_{j=1}^{\infty} 2(M_j+M_{j+1})$ shows that S' *converges uniformly over all sequences* σ, since $|u_j| \leqq 2 \max (M_j, M_{j+1}) \leqq 2(M_j+M_{j+1})$.

Further, each term of S' approaches the corresponding term of I' as $\Delta \to 0$, by the hypothesis of Riemann integrability.

It follows that the series I' converges and that $I'=\lim_{\Delta \to 0} S'$, by a known theorem on uniformly convergent series.*

But the convergence of the series I' is equivalent to the convergence of the integral I, and $I=I'$. For, the difference between $\int_a^t f(x)dx$ and the partial sum of I' from $j=1$ to l, the largest integer such that $a+l-1<t$, is numerically not greater than M_l and $M_l \to 0$ as $t \to \infty$.

A similar argument shows that the convergence of S' is equivalent to the convergence of S and that $S'=S$.

Hence $I=\lim_{\Delta \to 0} S$ and the theorem is proved.

A NOTE ON JOINT VARIATION

R. A. Rosenbaum, Reed College

There is a small point in connection with the topic of variation and proportionality which frequently disturbs the beginner, for many texts do not treat it adequately. Some students are still bothered by it when they take a course in thermodynamics and have to deal with relations of p, v, and T.

The trouble seems to arise from text-book statements similar to the following: "z is said to vary jointly as x and y if $z=kxy$. It is evident that, if y is held

* See, for example: W. F. Osgood, Funktionentheorie, Berlin, vol. 1, 5th ed., 1928, pp. 620–621.

constant, z varies as x, and that, if x is held constant, z varies as y." So far everything is clear, but, when the student comes to work problems, he is expected to assume the converse of the above, that is, to use the theorem: "If z varies as x, and z varies as y, then $z=kxy$." That this theorem is indeed true may be made to seem reasonable to the beginner by replacing the elliptical statement "z varies as x and z varies as y" by the complete statement "z depends on both x and y. When y is held constant, z varies as x, and, when x is held constant, z varies as y." A proof of the theorem which may be given to college students is the following:

If z varies as x when y is held constant, then $z=Rx$. For a fixed y, R is a constant, but for a different y, R may have a different value. In other words, R is a function of y; $R=f_1(y)$, say. Then

$$z = f_1(y) \cdot x. \tag{1}$$

Similarly, if z varies as y when x is held constant,

$$z = f_2(x) \cdot y. \tag{2}$$

Dividing (1) by (2), we obtain:

$$\frac{f_1(y) \cdot x}{f_2(x) \cdot y} = 1$$

or

$$\frac{f_1(y)}{y} = \frac{f_2(x)}{x}. \tag{3}$$

Since the left-hand side of (3) is a function of y and the right-hand side is a function of x, and since the equality holds for all x and y, each side must be equal to some constant, k.

i.e.

$$\frac{f_1(y)}{y} = k = \frac{f_2(x)}{x}$$

or

$$f_1(y) = ky \tag{4}$$

and

$$f_2(x) = kx. \tag{5}$$

Substituting from (4) in (1) or from (5) in (2), we have

$$z = kxy.$$

A similar type of proof may be used when more than three variables are involved, or when not all the variation is direct.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Science and Sanity. Second Edition. By Alfred Korzybski. Lancaster, Pa., The Science Press Printing Company, Distributors, 1941. $6.00.

This second edition of Korzybski's book is a reprint of the first edition, reviewed in this MONTHLY, Nov. 1934, with a 35-page pamphlet of comments on, and extracts from reviews of, the first edition, a new introduction of 52 pages, and a bibliography of 100 books and 32 monographs. Titles as varied as S. Zuckerman's *The Social Life of Monkeys and Apes* and Hermann Weyl's *The Mathematical Way of Thinking*, may suggest some indication of the wide scope of general semantics as presented in *Science and Sanity*.

There is nothing to add to the notice of the first edition, except one general observation: any book that was ever worth reading has been cordially damned by at least two persons. With this in mind, the author may see fit to exhibit in his third edition a select anthology of the fatuous things that have been said about general semantics, and his contribution to it, in the eight years between the two editions. Such an exhibition would be more illuminating to serious students than a hundred pages of laudatory remarks.

E. T. BELL

Mathematical Monographs. Edited by D. R. Curtiss. Northwestern University Studies in Mathematics and the Physical Sciences, No. 1. Evanston, Northwestern University, 1941. 7+172 pages. $2.25.

Maxima and Minima of Functions of Two or More Variables. By D. R. Curtiss.

The problem is the determination of proper extremes of a function $f(x_1, \cdots, x_n)$ which is real, single valued, and of class $C^{(r)}$ in a neighborhood of the origin. Taylor's expansion for the function uses the polynomials

$$G_m = \frac{1}{m!}\left(x_1\frac{\partial}{\partial x_1} + \cdots x_n\frac{\partial}{\partial x_n}\right)^m f(0, \cdots 0) = \sum c_{ij\cdots} x_1^i x_2^j \cdots,$$

where $i+j+\cdots=m$.

The first criteria are analogous to the well-known rules in case of $f(x)$. A careful analysis includes the usually slighted case in which the first non-identically vanishing G is semi-definite. The methods of Scheefer, Stolz, and Von Dantscher are reviewed with application to the semi-definite case.

The next method considers the problem of an extreme of $f(x, y)$ at the origin as a problem in conditioned extremes, namely that of an extreme of $f(x, y)$ on the curve $\partial f/\partial y = 0$. The result is expressed in terms of a set of poly-

nomials in the coefficients of the G_m. The process is a simplification of that of Stolz.

A new and most unusual process starts with the preparation theorem of Weierstrass which replaces $f(x, y)$ by $F(x, y)$, a "pseudo polynomial" in y. Then $f(x, y)$ has a proper extreme at the origin if no real branches of $F=0$ go through the origin. To test for this possibility the theorems of separation for roots of an algebraic equation are applied to $F(x, y)$ with x sufficiently small. Finally, Sturm's functions are derived directly from $f(x, y)$ by a new algorithm of greatest common divisor.

A well organized group of problems is included to show the advantages of the various methods.

The Statistics of Time Series. By H. T. Davis.

Some of the mathematical problems are presented which have arisen in the recent development of statistics in the direction of dynamics.

The important connections between the theory of integral equations and approximations by least squares is followed by an analysis of the theory of multiple regressions from the point of view of integral equations. These relations are used to establish a general significance test for the coefficients of an empirically determined regression. This followed the introduction of Schuster's periodogram $R(T)$ and spectrum of $y(t)$ in the harmonic analysis of time series, and the investigation of the distribution of the values of $R^2(T)$ in the test of the significance of maxima in $R(T)$.

The auto-correlation function, a special case of serial correlation, is related to Schuster's periodogram. The spectrum of a statistical time series can be obtained from the spectrum of a mathematical function that generates the same serial correlation function. A number of mathematical formulations of hysteresis preceded the new one in which serial correlation analysis is used. An example illustrates the process.

In the theory of random series an investigation is carried out of the effects of linear operators upon a series of random numbers; and an analogue of the random series is given in continuous variables.

The problem of the dynamics of economic time series was simplified by considering only the price series. To remedy some defects of previous assumptions a new formulation of the dynamic problem requires the maximum of an integral whose integrand depends on the utility function, a function that reflects shocks of war, droughts, etc., and a function that depends on surpluses. Statistical verification of this theory is satisfactory.

Topics in Continued Fractions and Summability. By H. L. Garabedian and H. S. Wall.

Connections have been known for some time between the Stieltjes moment problem and continued fractions and between the moment problem for the interval (0, 1) and the theory of summability by matrices of finite reference.

Now the Stieltjes continued fraction has been related to the theory of summability. Some of the results are studied and presented in inspiring form.

It is assumed that the methods of summation will have finite and regular matrices of reference. The Hausdorff matrix depends on the sequence $\{c_n\}$, defined by Stieltjes integrals $c_n = \int_0^1 u^n d\phi(u)$, $n = 0, 1, 2, \cdots$, where the mass function $\phi(u)$ determines the regularity of the matrix. The Hausdorff theory includes that of Cesàro, Hölder, and Euler-Knopp as special cases and extends their range of definition.

Various methods are shown by which inclusion problems in the domain of Hausdorff matrices may be studied. Sequences from row, column, or diagonal of the difference matrix Δ are essentially regular depending on the continuity at 0 and at 1 of the mass function of the base sequence. Inclusion problems are difficult when connected with the matrix Δ where the Hölder method of summation is associated with the base sequence. One special proof uses an integral equation technique to show that a moment sequence is regular and exhibits the mass function for it.

The Stieltjes moment sequence $\{c_n\}$ depends on integrals $c_n = \int_0^\infty u^n d\phi(u)$ where $\phi(u)$ is a bounded increasing function with an infinite number of points of increase. Stieltjes showed how a continued fraction determines a Stieltjes moment sequence, and conversely. When the range on the integral is (0, 1), totally monotone sequences $\{c_n\}$ are those for which the integrals hold for some monotone increasing function $\phi(u)$. The exact form of the continued fraction is found for this case and then a complete correlation exists between totally monotone sequences and continued fractions. This problem brought to light an important relation for continued fractions. A number of applications are considered.

A formulation is given of new conditions for regularity of totally monotone sequences in terms of the corresponding continued fraction. The older theorem has been provided with a new proof. A splendid group of problems includes periodic continued fractions and examples of moment sequences generated by continued fractions.

In conclusion, a class of Hausdorff methods of summation is studied with respect to its effectiveness in the analytic continuation of a power series $\sum a_n z^n$ outside its circle of convergence.

Spectra of Quadratic Forms in Infinitely Many Variables. By E. D. Hellinger.

For a class of quadratic forms in infinitely many variables, Hilbert discovered properties partly analogous to known theories but also he encountered a new phenomenon. The object here is to outline the development of this theory by means of examples.

A short resumé of some theorems of linear algebra and analytic geometry of n-dimensions includes analytical methods for deducing the theorems and applications of the theory. Then the investigation is extended to real quadratic forms in infinitely many variables of Hilbert space. For completely continuous

forms the statements about algebraic forms are virtually repeated except there is a countable infinite of characteristic values which converge to zero. This set is called the spectrum.

The Jacobi form has a continuous spectrum, $-1 \leqq \lambda \leqq 1$. Other results are analogous with such changes as sums replaced by integrals. Quadratic forms are constructed that have a segment $(-1, 1)$ as a multiple spectrum or that have both continuous and discontinuous spectra. Finally, a connection is made with continued fractions.

Bounded quadratic forms in infinitely many variables possess analogous properties. The preceding forms are bounded but are not sufficient to build all bounded forms. A suitable generalization, using Hellinger integrals, gives a new form for which all statements can be generalized. This form has a continuous spectrum, consisting of that perfect set on which the basis function increases. One example has a spectrum which is a perfect set, nowhere dense in the interval $(0, 1)$. A bounded form is found whose spectrum is continuous in the interval $(-1, 1)$ but which can not be transformed orthogonally into the Jacobi form. Conditions are discussed under which forms are combined into equivalent forms. One method is given for constructing spectrum and basis function of a given bounded quadratic form.

In case of non-bounded quadratic forms only some properties are similar to the preceding cases. A form is given whose spectrum covers the whole real axis and another where the spectrum is not uniquely determined. This indicates only a few of the differences.

MARIE M. JOHNSON

Supplement to Pandiagonal Magic Squares of Prime Order. By A. L. Candy, Author and Publisher. 1003 H Street, Lincoln, Nebraska, 1942. 3+30 pages.

In the first twenty-two pages of this pamphlet the author has shown how to extend considerably the number of pandiagonal magic squares of prime order of the type which fall under his Class II. Squares of this sort were treated at length in his book, *Pandiagonal Magic Squares of Prime Order*, reviewed in this MONTHLY, vol. 47, 1940, p. 563.

The remaining pages of the pamphlet are supplementary to the last section of the author's book, *Pandiagonal Magic Squares of Composite Order*, reviewed in this MONTHLY, vol. 48, 1941, p. 628.

G. E. RAYNOR

NEW BOOKS RECEIVED

Aircraft Mathematics. By S. A. Walling and J. C. Hill. Cambridge, University Press; New York, Macmillan Co., 1942. 189 pages. $1.25.

Analytic Geometry. By C. H. Lehmann. New York, John Wiley and Sons; London, Chapman and Hall, Ltd., 1942. 14+425 pages.

The Gist of Mathematics. By J. H. Moore and J. O. Mira. New York, Prentice-Hall, Inc., 1942. 12+726 pages. $5.35.

Differential Equations. Revised edition. By M. Morris and O. E. Brown. New York, Prentice-Hall, Inc., 1942. 11+355 pages. $3.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

THE LINEAR GRAPH

M. C. SHOLANDER, Brown University

A much overlooked source of topics for mathematics clubs is the linear graph. An important branch of combinatorial topology, this theory seems to be best known as a tool for solving problems of the recreational type.* It has found, however, fundamental applications in the study of electrical systems, chemistry, logic, curve theory, and the theory of invariants.

A linear graph is a set of elements, called arcs and points, which must satisfy only the condition that each arc is associated with two points, called its end points. In most cases it may be conveniently pictured in the plane by means of diagrams such as those found in this paper. Linear graphs which have only a finite number of elements are called finite. The degree of a point is defined as the number of arcs for which that point serves as end point. An end point of first degree will be called a free end point and the arc associated with a free end point will be called a free arc. A representative theorem at this stage is, "The number of points of odd degree in a graph is even."

Euler is credited with initiating the theory with his study of the problem of the bridges of Königsberg and allied problems. This problem—to traverse the bridges in the neighborhood of the city, each once and only once, in such a way as to return to one's starting point—may be pictured graphically by taking the land areas (islands and river banks) as points and the bridges as arcs. The resulting graph (Fig. 1) may easily be checked to show that the problem has no solution. This led Euler to the formulation of a general problem—to find for what linear graphs there exists a path (called the Euler line) which traverses each arc once and only once and returns to its starting point. The solution was soon found. A graph has an Euler line if and only if it is finite, connected, and such that all its points are of even degree. This result has immediate application to many recreation problems. A simple illustration is afforded by the domino problem: to place all the dominos of a set in a closed circular array such as (01) (12) · · · (40). The trick is to represent each domino by an arc whose end points are

* Recreational problems referred to in this paper may be found in W. W. R. Ball, Mathematical Recreations and Essays, eleventh edition, 1939; namely, The Bridges of Königsberg, p. 242; The Domino Problem, p. 250; The Knight's Tour, p. 174; The Four-color Problem, p. 222; The Labyrinth Problem, p. 254.

For the consideration of these and other problems as applications of linear graph theory see Dénes König, Theorie der Endlichen und Unendlichen Graphen, Leipzig, 1936. This paper borrows material freely from the latter book.

numbered according to the values of the domino's halves. We see that the problem may be solved for a "double-n" set of dominos if and only if n is even.

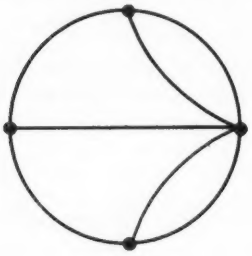
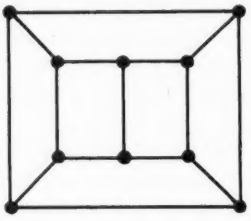


FIG. 1.



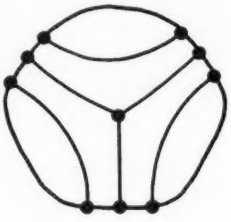


FIG. 2.

A problem seemingly of equivalent difficulty, and yet one which is as yet unsolved, is to find the necessary and sufficient conditions that a graph has a Hamiltonian line. This is a line which passes through each point of the graph once and only once and returns to its starting point. The graph of Figure 1 has such a line. The problem of the Knight's Tour may be shown to be one which requires for its solution the Hamiltonian line of a certain linear graph composed of 64 points and 168 properly placed arcs. An interesting application of this line can be made for the currently popular "four-color problem." A sufficient condition that a "map" is "colorable" is that the linear graph which it determines has a Hamiltonian line. Figure 2 will illustrate the difficulty of the problem. We have two graphs, each with 10 points and 15 arcs, only one of which admits a Hamiltonian line.

The "Labyrinth Problem" is one which has been more successfully attacked. We have space here only to remark that it may be best formulated as a problem in linear graph theory and that it, in itself, is an excellent topic for a complete mathematics club talk.

The fact that in organic chemistry we commonly picture molecules as linear graphs with atoms as points and bonds between atoms as arcs leads to other applications of linear graph theory. As background for a typical problem we make the following definitions. A "cycle" is defined as a closed circuit of arcs which passes through no point more than once—*e.g.*, $(A_1A_2)(A_2A_3)(A_3A_1)$ where A_1, A_2, and A_3 are distinct. A "tree" is a finite connected graph which contains no cycles. We shall say that the connectivity of a graph is k if it is necessary to remove k arcs from the graph in order to make a tree of it. Thus a tree has connectivity 0 and a cycle has connectivity 1. We can prove that a graph G with a_0 points and a_1 arcs has the connectivity a_1-a_0+1.

Now suppose we wish to determine all possible ways of picturing a molecule C_nH_{2n+2} in the paraffin series. Each hydrogen atom is assumed to have a single bond, so we must associate it with a free end point. The carbon atoms (having

valence four) correspond to points of degree four. We compute $a_0 = n + (2n+2) = 3n+2$, $a_1 = \frac{1}{2}(4n+2n+2) = 3n+1$, and deduce that the connectivity of the molecule is 0. Hence the molecule is a tree and there is no possibility of cycles formed by the bonds. For example, no two carbon atoms may be linked by multiple bonds. Each type of tree formed from n points yields a possible carbon configuration on which the hydrogen atoms may be "hung." No other configurations are possible.

Finally, there exist problems which because of the complexity of their conditions do not lend themselves to solution by known theorems of linear graphs, and yet whose solution may be facilitated by simply presenting the problem in graphical form. We illustrate this with the "Problem of the Missionaries and Cannibals." It is required to take three missionaries and three cannibals across a river by means of a boat which holds at most two persons. The missionaries and one of the cannibals can row. The situation is complicated by the requirement that on neither bank is it ever permitted for the cannibals to outnumber the missionaries.

To picture the problem we denote any missionary by M, the rowing cannibal by K, and either of the other two cannibals by C. A point of the graph will be assigned to any permissible grouping of the party on the first bank of the river. Specifically, we have the sixteen points:

1. $MMMCCK$
2. $MMMCC$
3. $MMMCK$
4. $MMMC$
5. $MMMK$
6. $MMCC$
7. $MMCK$
8. MMM
9. CCK
10. MC
11. MK
12. CC
13. CK
14. C
15. K
16. 0,

where 0 indicates the absence of everyone. We now place arcs on the graph between a point and any other which may result from the first by either the arrival or departure of the boat—*e.g.* between 1 and 4. This yields 25 arcs (Fig. 3) and our problem is reduced to that of going from point 1 to point 16 by means of a proper path. We have so numbered our points that—due to the boat's alternate departure and arrival—the proper path is one which zig-zags from lower to higher numbers, then from higher to lower. It is now a simple matter to inspect the graph and produce one of the few possible solutions. Such a solution is 1, 4, 3, 8, 5, 11, 7, 10, 6, 12, 9, 14, 13, 16.

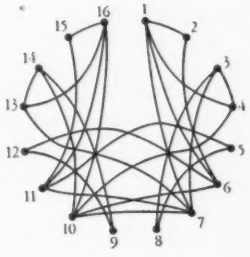


FIG. 3.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 536. *Proposed by Norman Miller, Queen's University.*

In a smooth hemispherical bowl of radius a, a needle of length $2l (l<a)$ is placed with one end projecting over the rim and is then released. Show that the needle will come to rest in a horizontal position or an inclined position according as l is less than or greater than

$$a\sqrt{\frac{2}{3}}$$

E 537. *Proposed by V. Thébault, San Sebastián, Spain.*

Let L, M, N and L', M', N' be the orthogonal projections of a point P on the sides and corresponding altitudes of a given triangle. Show that the lines LL', MM', NN' are in general concurrent, and find the locus of P when they are parallel.

E 538. *Proposed by R. V. Heath, Wall St., New York City.*

Find a perfect square whose digits form one of the permutations of eight consecutive digits. (Cf. E 532.)

E 539. *Proposed by Howard Eves, Chattanooga, Tennessee.*

Give a ruler construction for finding the centers of three given linearly independent circles, no two of which are intersecting, tangent, or concentric.

E 540. *Proposed by L. M. Kelly, U. S. Coast Guard Academy.*

Can the radius of the sixteen-point sphere ever be one-half the circumradius of the tetrahedron? (The sixteen-point sphere passes through the circumcenters of the faces.)

SOLUTIONS

The Round Table

E 500 [1941, 699]. *Proposed by S. H. Gould, University of Toronto*

In how many ways can p gentlemen and q ladies sit at a circular table if each lady has the choice, as long as gentlemen are still available, of sitting on a chair or on a gentleman's knees?

Solution by W. R. Van Voorhis, Fenn College, Cleveland, Ohio

Let m be the number of ladies who elect to sit on gentlemen's knees. Then the p gentlemen and $q-m$ disdainful ladies can take chairs in $(p+q-m-1)!$ ways. The complaisant ladies may be chosen in $\binom{q}{m}$ ways. There are p gentlemen's knees available for the first of these ladies, $p-1$ for the second, and so on. Hence they can all be seated in

$$(p+q-m-1)!\binom{q}{m}p(p-1)\cdots(p-m+1)$$

ways. Summing for the possible values of m, we find the total number of ways to be

$$\sum_{m=0}^{\min(p,q)}\binom{p}{m}\binom{q}{m}m!(p+q-m-1)!.$$

Also solved by the proposer.

Editorial Note. For the corresponding problem when the gentlemen are kept fixed, we must divide the answer by $(p-1)!$. It can then be expressed as

$$q!\sum_{m=0}^{\min(p,q)}\binom{p}{m}\binom{p+q-m-1}{p-1},$$

which is $q!$ times the coefficient of x^q in the expansion of

$$\left(\frac{1+x}{1-x}\right)^p.$$

The case where $p=q=3$ is relevant to the theory of sense in the projective line. See p. 32 of Coxeter, *Non-Euclidean Geometry*, Toronto, 1942.

The Generalized Simson Line

E 501 [1942, 61]. *Proposed by Daniel Arany, Budapest, Hungary.*

If A, B, C, I, J, X are six points on a conic, while L, M, N are points on the respective sides BC, CA, AB of the triangle ABC, and if further the three pencils $L(BXIJ)$, $M(CXIJ)$, $N(AXIJ)$ are projectively related, prove that the points L, M, N are collinear.

I. *Solution by L. M. Kelly, U. S. Coast Guard Academy.*

The notation suggests projecting I and J into the circular points at infinity. Then the conic goes into a circle, and the projected theorem is as follows:

If from any point X on the circumcircle of the triangle ABC lines be drawn making equal angles with the respective sides, the three "feet" of such lines will be collinear.

This is a well known generalization of the Simson line property, and can easily be proved by considering cyclic quadrangles.

Also solved thus by Howard Eves.

II. *Solution by R. L. Wright, University of Toronto.*

The nine points $A, B, C, I, J, X, L, M, N$ lie on two distinct cubics: one consisting of the conic $LMCXIJ$ and line ABN, another consisting of the conic $NLBXIJ$ and line CAM. These are therefore nine *associated* points (such that any cubic through eight of them goes also through the ninth). The conic $ABCIJX$ and line LM form a cubic through the first eight points, and so also through N. But $ABCIJX$ does not contain N; therefore LM does.

A Number and its Fourth Power

E 502 [1942, 61]. *Proposed by V. Thébault, San Sebastián, Spain.*

Find a number and its fourth power which together have nine digits, all different.

Solution by W. E. Buker, Pittsburgh Public Schools.

If n and n^4 together consist of nine digits, we must have $31 < n < 57$. Furthermore, n cannot end in 0, 1, 5, or 6, as then the ending of n^4 would repeat. Noting these restrictions, and leaving out 33, 44, 55, we try the following thirteen possibilities by the aid of a table of squares:

$$32, 34, 37, 38, 39, 42, 43, 47, 48, 49, 52, 53, 54.$$

We find that n and n^4 duplicate digits unless $n = 32$, in which case $n^4 = 1048576$.

We note that "casting out nines," usually of service in such problems, fails us here, as we do not know which digit is omitted.

Also solved by R. K. Allen, Paul Brock, M. L. Constable, S. T. Eriksson, Howard Eves, R. E. Greenwood, E. P. Starke, and the proposer.

The Centroid of a Tetrahedron

E 503 [1942, 61]. *Proposed by N. A. Court, University of Oklahoma.*

Through a point M lines are drawn meeting the pairs of opposite edges of a given tetrahedron in the pairs of points U, X; V, Y; W, Z. Prove that if M bisects each of the three segments UX, VY, WZ, it coincides with the centroid of the tetrahedron.

Solution by Howard Eves, Syracuse University.

Let $ABCD$ be the tetrahedron, with X on BC, Y on CA, Z on AB, U on DA, V on DB, W on DC. Since M bisects UX, VY, WZ, it follows that VW and YZ WU and ZX, UV and XY, are parallel. Therefore YZ and BC, ZX and CA, XY and AB, are parallel. But this is possible if and only if X, Y, Z are the midpoints of the edges BC, CA, AB. Similarly U, V, W are the midpoints of the edges on which they lie. Thus M is the centroid of the tetrahedron.

Also solved by Albert Furman, L. M. Kelly, W. T. Short, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4050. *Proposed by Arnold Dresden, Swarthmore College.*

If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that no two differ by a multiple of π, prove that

$$\sum_{k=1}^{n} \prod_{i=1, i\neq k}^{n} \cot (a_k - a_i) = \sin \frac{n\pi}{2}.$$

4051. *Proposed by Arnold Dresden, Swarthmore College.*

If $a_1, a_2, \cdots, a_n$ are n distinct complex numbers, $n>1$, such that none of them and none of the differences is a multiple of π, show that

$$\sum_{j=1}^{n} \prod_{i=1, i\neq j}^{n} \cot (a_j - a_i) \cot a_j + (-1)^n \prod_{i=1}^{n} \cot a_i = \sin \frac{(n+1)\pi}{2}.$$

4052. *Proposed by H. S. Wall, Northwestern University.*

If $\frac{1}{2}<r\leqq 1$ and $|z| \leqq (2r-1)/4r^2$, prove that

$$\left| e^z - \frac{4r^2}{4r-1} \right| < \frac{2r(2r-1)}{4r-1}.$$

4053. *Proposed by E. P. Starke, Rutgers University.*

Show that all triangles inscribed in an ellipse and having their centroids at the center of the ellipse have the same area, which is the greatest possible area for an inscribed triangle.

Show that all triangles circumscribed about an ellipse and having their centroids at the center of the ellipse have the same area, which is the least possible area for a circumscribed triangle.

4054. *Proposed by V. Thébault, San Sebastian, Spain*

Find the base less than 100 for which the number 2101 is a perfect square.

4055. *Proposed by V. Thébault, San Sebastian, Spain.*

Find two numbers of ten digits of the form *aabbccddee* such that one is a perfect square and the other is a perfect square increased by 7.

SOLUTIONS

Quadrilateral and Orthopoles

3991 [1941, 214]. *Proposed by V. Thébault, San Sebastián, Spain*

Four straight lines Δ_i in a plane determine a complete quadrilateral (Q) forming four triangles $T_1 \equiv (\Delta_2, \Delta_3, \Delta_4)$, $T_2 \equiv (\Delta_1, \Delta_3, \Delta_4)$, *etc.*, with the orthocenters H_i. Show that the orthopoles of the straight line $\Delta \equiv (H_1, H_2, H_3, H_4)$, with respect to the four triangles, of the parallels to Δ_i through H_i are the orthogonal projections of the Miquel point on the sides of (Q).

Solution by J. W. Clawson, Ursinus College

It is well known that the circumcenters, C_i, of the triangles, T_i, are concyclic and that the Miquel (or focal) point of (Q) lies on this circle. We take this circumcentric circle as the unit circle in the complex plane and take the Miquel (focal) point, F, to be the unit point on the axis of reals. Taking the centers, C_i, to be turns, t_i, on the unit circle, it is not difficult to show that the side Δ_i has the equation $z - \bar{z} \cdot t_j t_k t_l = t_j + t_k + t_l - t_{ijl} - t_i t_k - t_i t_l$. In this equation z is the complex number of a point on the side and $\bar{z}$ the conjugate complex number. The equation was derived by finding the point of intersection, other than F, of the circles with centers C_j, C_k passing through F. This gives a vertex of (T), A_{il}, to be $t_j + t_k - t_j t_k$. The line Δ_i is determined by A_{il} and A_{ik} (or A_{ij}).

The foot of the perpendicular from F to Δ_i is

$$\tfrac{1}{2}(1 + t_j + t_k + t_l - t_j t_k - t_j t_l - t_k t_l + t_j t_k t_l).$$

Incidentally, these four feet lie on the pedal line, $z + \bar{z} \cdot s_4 = \tfrac{1}{2}(1 + s_1 - s_2 + s_3)$, where s_m is the sum of the products m at a time of t_1, t_2, t_3 and t_4.

Again, H_i is $s_1 - t_i(t_j + t_k + t_l)$. Hence Δ_i', the parallel through H_i to Δ_i, is

$$z - t_j t_k t_l \cdot \bar{z} = s_1 - s_2 + (t_j t_k + t_j t_l + t_k t_l)/t_i.$$

Solving two of these equations simultaneously, we find that A_{jk}', one of the vertices of triangle T_i' determined by Δ_j', Δ_k', Δ_l', is $t_j + t_k + t_i t_l - s_2$.

Again, the four points H_i lie on Δ, which is $z + s_4 \cdot \bar{z} = s_1 - s_2 + s_3$, the orthocentric line of (Q).

To find the orthopole of Δ with respect to T_i', we find the feet of perpendiculars from A_{jk}' and A_{jl}' (or A_{kl}') on Δ, and then the equations of lines through these points perpendicular to the opposite sides of triangle T_i'. These perpendiculars intersect in the orthopole. The algebra is somewhat laborious but leads to the same point found above as the foot of the perpendicular from F to Δ_i.

Editorial Note. A solution is easily obtained by using some of the results in the solution of 3839 [1939, 604], and additional simple computations. In the above notation we have the following:

$$\Delta_i\colon\ m_iy - x - am_i^2 = 0; \qquad A_{ij}\colon\ am_im_j,\ a(m_i + m_j);$$
$$H_i\colon\ -a,\ a(\sigma_1^i + \sigma_3^i); \qquad F\colon\ a, 0; \qquad \Delta\colon\ x + a = 0;$$
$$\Delta_i'\colon\ m_iy - x - a(1 + \sigma_2 + \sigma_4 - \sigma_2^i) = 0;$$
$$A_{ij}'\colon\ -a(1 + \sigma_4 + m_im_j),\ a(m_k + m_l);$$

where σ_t is the elementary symmetric function of the four m's and σ_t' of three m's after omitting m_j. The projection of A_{ij}' on Δ is the point $-a,\ a(m_k+m_l)$; and the equation of the straight line through this point perpendicular to Δ_k' is $y+m_kx-am_l=0$. Hence the orthopole of Δ with respect to T_l' is $0,\ am_l$, and this point is the projection of F on Δ_l.

The quadrilaterals (Q) and (Q') are symmetric with respect to the point $-a(1+\sigma_4)/2,\ a\sigma_1/2$, and this center of symmetry lies on $y=a\sigma_1/2$, the common Newton line for (Q) and (Q').

LEGENDRE TRANSFORMATION

4000 [1941, 409]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Given the functions $y=f(x)$, $Y=F(X)$, deduce from the transformation formulas

$$\frac{y'}{aX + bY + d} = \frac{-1}{bX + cY + e} = \frac{y - xy'}{dX + eY + f},$$

the following:

$$\frac{Y'}{ax + by + d} = \frac{-1}{bx + cy + e} = \frac{Y - XY'}{dx + ey + f},$$

where a, b, c, d, e, f are arbitrary constants. Give a geometric interpretation; and deduce from it the Legendre transformation.

Solution by Peter Chiarulli, Student, Brooklyn College.

From the given equality of three fractions we deduce from the equality of the first and third fractions that each is equal to

$$y/[X(ax + d) + Y(bx + e) + (dx + f)]; \tag{1}$$

and from the equality of this with the second given fraction we find that

$$X(ax + by + d) + Y(bx + cy + e) + (dx + ey + f) = 0. \tag{2}$$

Differentiation of (2) with respect to X gives

$$(ax + by + d) + Y'(bx + cy + e) + \frac{dx}{dX}[(aX + bY + d) + y'(bX + cY + e)] = 0.$$

Since the factor of dx/dX is zero by the given equality of the first and second fraction, we have

$$\frac{Y'}{ax + by + d} = \frac{-1}{bx + cy + e} = \frac{Y}{X(ax + by + d) + (dx + ey + f)},$$

where the third fraction results from (2). This gives the desired result by a combination of the first and third fraction in these derived equalities.

If $a=-1$, $e=1$, $b=c=d=f=0$, we have the Legendre transformations

$$y' = X, \quad xy' - y = Y; \quad \text{and} \quad Y' = x, \quad XY' - Y = y.$$

The line (2) is the polar of the point (x, y) with respect to the conic

$$\text{(A)} \qquad ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0;$$

and we now consider the polars of points on the curve $y=f(x)$ with respect to the conic (A), and the envelope $Y=F(X)$ of these polars. Regarding x as the variable parameter we find by differentiation of (2).

$$y'(bX + cY + e) + (aX + bY + d) = 0.$$

Using the above process of combination of fractions on this last result and (2) we obtain the given set of equalities. Therefore the geometric interpretation is that, if we have two curves $y=f(x)$ and $Y=F(X)$ connected by the given transformation formulas of the problem, then each curve is the envelope of the polars of points on the other curve with respect to the conic (A).

POSTPONEMENT OF THE PUTNAM COMPETITION

The William Lowell Putnam Mathematical Competition has been postponed for the present by agreement of the Putnam Trustees and the Board of Governors of the Mathematical Association. This is occasioned mainly by the preoccupation of both teachers and students with war courses in mathematics.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Associations, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, State College, Pa., Oct. 23–24, 1942
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Baltimore, Dec. 5, 1942
METROPOLITAN NEW YORK
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City, Dec. 4, 1942
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, Jan. 30, 1943
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE, fall, 1942
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